On conformal equivalence of Riemann-Finsler metrics and special Finsler Manifolds

Doktori (Ph.D.) értekezés

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Debreceni Egyetem

Debrecen, 2000

Ezen értekezést a Debreceni Egyetem Matematikai doktori program Differenciálgeometria és alkalmazásai alprogramja keretében készítettem 1996–1999 között és ezúton benyújtom a Debreceni Egyetem doktori Ph.D. fokozatának elnyerése céljából.

Debrecen, 2000

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Debrecen, 2000

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Köszönetnyilvánítás

A szerző ezúton mond köszönetet Dr. Szilasi Józsefnek lelkiismeretes témavezetői munkájáért. Közreműködése és tanácsai nélkül ez a disszertáció nem készülhetett volna el.

Acknowledgement

The author would like to thank Dr. József Szilasi, his supervisor, for his invaluable help and encouragement during the years of Ph.D. course. We are also grateful to him for his suggestions and contribution to preparation of this Ph.D. dissertation.

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INTRODUCTION

In this Ph.D. dissertation we undertake a quite comprehensive survey of general theoretical elements of Finsler geometry. The primary aim of this survey is to present a standard system of notations and terminology built on three pillars: the theory of horizontal endomorphisms, the calculus of vector-valued forms and a "tangent bundle version" of the method of moving frames. On the other hand we present a systematic treatment of some distinguished Finsler connections and some special Finsler manifolds. In particular we are interested in the conformal theory of Riemann-Finsler metrics and the theory of Wagner connections and Wagner manifolds. As we shall see, they are closely related. Finally, we investigate a special conformal change of the metric proving that its existence implies the Finsler manifold to be Riemannian. (The necessity is clear.)

I

By the classical conception of Finsler geometry the geometrical objects are characterized in terms of local coordinate systems. In exposition and application of such a theory of Finsler manifolds the tools of tensor calculus, adapted to the Finsler setting play a dominant role. In Matsumoto's fundamental work [24] the starting point of this adaptation is a principal bundle, the so-called Finsler bundle constituting the area where problems of Finsler geometry, mainly of Finsler connections are presented: "Differential-geometric concepts and quantities are mainly introduced in the principal bundle over the tangent bundle induced from the linear frame bundle ... " (see [24], p. 46). As it can be seen, Matsumoto's theory of Finsler connections is developed from a strongly generalized standpoint. Here we follow an alternative approach based on Grifone's theory of nonlinear connections [10], [11] (whose role is played in our presentation by the so-called *horizontal endomorphisms*) and the coordinate-free, "intrinsic" calculus of the vector-valued differential forms and derivations established by A. Frölicher and A. Nijenhuis [9]. Our main purpose is to insert the theory of Finsler connections and the foundations of special Finsler manifolds in the new approach of Finsler geometry. The first epoch-making steps in this direction were done by J. Grifone himself, our work can be considered as a systematic continuation of the program initiated by him. Technically, we enlarged and – at the same time – simplified the apparatus by using the tools of tangent bundle differential geometry. This means first of all the consistent use of a special frame field, constituted by vertically and completely (or vertically and horizontally) lifted vector fields. Thus the third pillar of our approach is the method of moving frames. It has a decisive superiority in calculations over coordinate methods: the formulation of the concepts and results becomes perfectly transparent, and the proofs have a purely intrinsic character. Nevertheless, coordinate-calculations will not be avoided completely. In section I/6 we felt that an indication of the connections to the traditional theory will be useful, while in sections I/7 and III/2 the nature of the problem forced us to use a suitable coordinate-system. Finally, we have to emphasise that the classical sources H. Rund [30], M. Matsumoto

[24] and, especially Hashiguchi's works [13], [14] etc. were indispensable and very stimulating for our investigations.

The dissertation is divided into three parts. In part I first of all we present a quite detailed exposition of the conceptual and calculational background. Although it means a practical summary (the troublesome details will be omitted) it seems to be enough to make our work self-contained as far as possible. As the next step we come to the overview of the fundamental facts and constructions concerning a Finsler manifold. Simply put a Finsler structure on a differentiable manifold M is a function

$$E : TM \to R$$

satisfying some conditions of differentiability, homogeneity and regularity. In conformity with the demands of Finsler geometry, the smoothness is not required or assured a priori on the whole tangent manifold TM. (It is well-known that Finsler structures without singularities are just Riemannian). With the help of the so-called *energy function* E we can introduce a (pseudo-) Riemannian metric

$$g : v \in TM \setminus \{0\} \to g_v, \quad g_v : T^v{}_vTM \times T^v{}_vTM \to R$$

(the so-called *Riemann-Finsler metric*) on the vertical subbundle of the "tangent bundle" TTM. This means that in such the geometry all of objects depend on both "position" and "direction". As it is usual in case of a Riemannian manifold we also can associate a canonical horizontal endomorphism (the so-called *Barthel endomorphism*) to the function E together with lots of important tensors and further geometrical structures such as Cartan tensors and the canonical almost complex structure on the Finsler manifold. Among others we pay a particular attention to the so-called *first* and *second Cartan tensors*. The second Cartan tensor is introduced in a more general situation as usual. It means that this tensor is associated to an arbitrary horizontal endomorphism instead of just the canonical one. In particular we investigate the connection between the symmetry properties of the tensor and the characteristic data of the horizontal endomorphism. Although our results belong to the foundations, they are new. The reason of this careful investigation is that the first and second Cartan tensors play an essential role in Finsler geometry as it will be demonstrated in section I/4. Here we present an invariant and axiomatic description of three notable Finsler connections (linear connections associated to a nonlinear one with the help of some conditions of compatibility): the *Berwald, Cartan* and *Chern-Rund connections.* We hope that our approach helps in better understanding the role of the different axioms, and open a path for further, essential generalizations. Theorems are organized as follows: the first group of the axioms characterizes a unique Finsler connection allowing us to derive the explicit rules of calculations for the corresponding covariant derivatives. Adding further conditions to them the second group yields the characterization of the three classical Finsler connections.

The next step is to insert the foundations of special Finsler manifolds such as the so-called *Berwald* and *locally Minkowski manifolds* into the framework has been elaborated by then. To realize our plan we need several technical observations summarized under the title *Basic curvature identities*. In this section we derive some important (partly well-known) relations between the curvature data of the different

Finsler connections, these will be indispensable for a tensorial description of the special Finsler manifolds studied in the last two sections. Section I/6 concentrates the characterization of Berwald manifolds; some of them (e.g. 6.5 and 6.7) are new, and all of the proofs are original. We believe that the compact, elegant and efficient formulations presented here demonstrate the power of our approach. In the concluding section I/7 the key observation is given in Proposition 7.2; this provides a very simple proof of the classical characterization of locally Minkowski manifolds. Finally, we have to emphasize that these results, more precisely the analogous ones play an important role in the theory of Wagner connections and Wagner manifolds. Since they are generalizations of the usual concepts in Finsler geometry (such as the Cartan connection and the Berwald or locally Minkowski manifolds) we present some of proofs in sections II/4 and II/5 in a more general situation as well. In this consideration one of the most important results is a generalization of a classical theorem (see I.6.3), first formulated and proved intrinsically by J. G. Diaz [8]. It contains equivalent tensorial characterizations of the vanishing of the second Cartan tensor associated to the Barthel endomorphism, i.e. the characterizations of the so-called *Landsberg manifolds*. In his thesis [8] the author gives a coordinate-free proof of this theorem using several explicit relations between the classical Cartan tensors and curvatures (or their lowered tensors) of the Cartan connection. We managed to reduce the number of these relations to some of fundamental ones and the theorem is proved in generality of Wagner connections and Wagner manifolds in section II/5; see Proposition 5.4. *Techniques we need to discuss them are suitable to reproduce lots of classical results as well.* We found this observation very useful.

II

In part II we start with the definition of conformal equivalence of Riemann-Finsler metrics. This relation is formally the same as that in Riemannian geometry. Two Riemann-Finsler metrics g and $\widetilde{g}$ are said to be *conformally equivalent* if there exists a function $\varphi : TM \setminus \{0\} \to R^+$ such that

$$\widetilde{g} = \varphi g.$$

It is an immediate consequence of the definition that the so-called *scale* or *proportionality* function φ can be prolonged to a smooth function on the whole tangent bundle, actually it is constant on each tangent space TpM $(p \in M)$. We give a modern proof of this fameous observation due to M. S. Knebelman [19]. In sections II/1 and II/2 we also derive some important conformal invariants and transformation formulas, first of all a key formula describing the change of the canonical spray of a Finsler manifold under a conformal change of the metric. As a consequence, we get immediately, how the Barthel endomorphism is changing. Having these results, one can also describe the change of the Berwald and Cartan (and other) connections, etc. A complete summary can be found in Hashiguchi's paper [14] using the classical coordinate methods of calculation. In order to illustrate the problem we derive how the second Cartan tensors are related in case of conformal equivalence of Riemann-Finsler metrics. An application of our results is also given in this section: we present an intrinsic proof of the classical theorem which (roughly speaking) states that in case of a simultaneous conformal and projective change the scale function is constant, i.e. the conformal change must be homothetic (see [30], p. 226).

After these "preliminaries" and illustrations we define the notion of Wagner connections. Such kind of Finsler connections were first constructed and used by V. V. Wagner [40]. With the help of this seemingly strange connection Wagner introduced the notion of generalized Berwald manifolds (especially – in present day terminology – *Wagner manifolds*) and he showed that this class of special Finsler manifolds contains any two-dimensional Finsler manifold with cubic metric. The next important steps in the extension of the theory of Wagner connections and generalized Berwald manifolds were taken by M. Hashiguchi [13]. He successfully carried over Wagner's ideas to the arbitrary finite dimensional case, characterizing the Wagner connections by an elegant system of axioms (cf. section II/3). One of the most important observations, due to M. Hashiguchi and Y. Ichijyo [16] is that Wagner connections are at the heart of the theory of conformal change of Riemann-Finsler metrics. Among others it turned out that the class of Wagner manifolds is closed under a conformal change of the metric. These results confirm Matsumoto's remarkable principle: "there should be existing a *best* Finsler connection for every theory of Finsler spaces" (see [25]).

In this part we demonstrate that the Frölicher-Nijenhuis formalism provides a perfectly adequate conceptual and technical framework for the study even of such complicated objects as Wagner connections. Our intrinsically formulated and proved results not only cover the classical local results but give a much more precise and transparent picture and open new perspectives. For example, we calculate the tension, the weak and strong torsion of a so-called *Wagner endomorphism* (the "nonlinear part" of a Wagner connection), i.e. data determining uniquely a nonlinear connection by Grifone's theory. As one of the main results we conclude that the rules of calculation with respect to a Wagner connection are formally the same as those with respect to the classical Cartan connection. These investigations are realized with the help of a number of new (but more or less) technical observations and a fine analysis of the second Cartan tensor belonging to a Wagner endomorphism. Basic curvature identities concerning a Wagner connection, including Bianchi identities are also derived. Using these results an important classical theorem on Landsberg manifolds will be generalized in section II/5 (cf. remarks at the end of part I). Finally, after a new intrinsic definition as well as several tensorial characterizations of Wagner manifolds, we present an intrinsic formulation and coordinate-free proofs for Hashiguchi-Ichijyo's theorems. In their joint work [16] the authors have explored the significance of Wagner manifolds relating them to the conformal changes of Riemann-Finsler metrics. Simply put, for any conformal change of the metric we can construct a special Finsler connection, the so-called *Wagner connection* with the help of the scale function. We can say that a Wagner connection is a *Cartan connection with non-vanishing* $(h)h$*-torsion*; i.e. it is a *generalized Cartan connection*. (The $(h)h$-torsion has a special semisymmetric form; cf. section II/3; Definition 3.1.) Then Wagner manifolds can be introduced on the model of classical Berwald manifolds. This means that the Wagner endomorphism, i.e. the nonlinear part of the Wagner connection is induced by a linear connection on the underlying manifold M. (Or, equivalently, the Wagner endomorphism is smooth on the *whole* tangent manifold TM.) In his paper [14], Hashiguchi suggested and (in some sense!) solved the problem: under what conditions does a

Finsler manifold become conformal to a Berwald (or locally Minkowski) manifold. "These conditions were, however, given in terms of very complicated systems of differential equation, for which appropriate geometrical meanings have been wanted", he wrote a year later in [16]. As it was shown these "appropriate geometrical meanings" were hidden in the notion of Wagner manifolds, sketched by the following diagram:

$$
\begin{array}{ccc}
 & \text{Wagner manifolds (with integrable Wagner endomorphism)} & \\
 & \nwarrow \quad \nearrow & \\
 & \searrow \quad \swarrow & \\
 & \text{Conformal change of the metric} & \\
 & \nearrow \quad \nwarrow & \\
 & \swarrow \qquad \searrow & \\
\text{Berwald manifolds} & \dashrightarrow & \text{Locally Minkowski manifolds} \\
 & \text{Integrability condition} &
\end{array}
$$

Namely, in the classical terminology: "The condition that a Finsler space be conformal to a Berwald space is that the space becomes a Wagner space with respect to a gradient $\alpha_i(x)$" ([16], Theorem B.).

III

In part III we deal with a special conformal change of Riemann-Finsler metrics introduced by M. HASHIGUCHI [14]. The point of the so-called *C-conformality* is that we require the vanishing of one of conformal invariants described in section II/1; cf. Proposition 1.12 and III. 2.1. Under this hypothesis the gradient vector field of the scale function becomes independent of the "direction", i.e. it will be a vertically lifted vector field. (Vector fields with such a property is called *concurrent* too; see e.g. [14], [28] and [37].)

Consider a conformal change $\widetilde{g} = \varphi g$, where the scale function φ is given in the form

$$\varphi = \exp \circ \alpha^v;$$

$\alpha \in C^\infty(M)$, $\alpha^v = \alpha \circ \pi$. It is said to be a *C-conformal change* at a point $p \in M$ if the following conditions are satisfied:

(a) $\qquad \alpha$ is regular at the point $p \in M$,

(b) $\qquad [J, \operatorname{grad} \alpha^v] = 0,$

where J is the canonical vertical endomorphism (or, in equivalent terminology, the canonical almost tangent structure of the tangent bundle $\pi : TM \to M$). In his cited work [14] Hashiguchi proved for some special Finsler manifolds (in his terminology: two-dimensional spaces, C-reducible spaces, spaces with (α, β)-metric etc.) that the existence of a C-conformal change of the metric implies that the manifold is Riemannian (at least locally; cf. the condition (a)). Here we show that Hashiguchi's result is valid without any extra condition. In terms of our characterization this means that the vanishing of some conformal invariants, like the conformal invariant first Cartan tensor, can be interpreted as a sufficient condition for a Finsler manifold to be Riemannian. (The necessity is clear.) Our result is based on a usual, but relatively "rigid" definition of Finsler manifolds: the differentiability of the energy function is required at *all* nonzero tangent vector, i.e. there is *no* singularity except for the zero vectors of tangent spaces. The basic idea we use to prove our statement is an observation on homogeneous functions. (Actually, we generalize the following well-known fact: *if a function is homogeneous of degree 0 and it is continuous at the origin, then the function is constant.*) In other words, the main points are the *homogeneity* and *continuity* of the Riemann-Finsler metric along the gradient vector field of the scale function which depends only on the "position" in case of a C-conformal change. Weakening the condition of differentiability new perspectives open to investigate the C-conformality. As an illustration we shall cite some valuable fragments from Hashiguchi's original ideas in one of the last remarks.

I. A NEW LOOK AT FINSLER CONNECTIONS AND SPECIAL FINSLER MANIFOLDS

1. Notation and some basic facts

1.1. M is a connected, paracompact, smooth (i.e., C^∞) manifold of dimension n, where $n \in \mathbb{N} \setminus \{0,1\}$. $C^\infty(M)$ is the ring of real-valued smooth functions on M, the $C^\infty(M)$-module of *vector fields* on M is denoted by $\mathfrak{X}(M)$.

1.2. $\forall k \in \{0,\dots,n\}$: $\Omega^k(M)$ is the module of *differential k-forms* on M; by convention $\Omega^0(M) := C^\infty(M)$. $\Omega(M) := \oplus_{i=0}^n \Omega^k(M)$ is the graded algebra of differential forms with multiplication given by the wedge product. To each vector field $X \in \mathfrak{X}(M)$ correspond two derivations of $\Omega(M)$: *the substitution operator* $\imath_X$ of degree -1, and the *Lie derivative* $\mathcal{L}_X$, of degree 0. These are related to the operator d of the *exterior derivative* through H. Cartan's magic formula

$$\mathcal{L}_X = [\imath_X, d] := \imath_X \circ d + d \circ \imath_X.$$

1.3. A *vector k-form* on M is a skew-symmetric k-multilinear map $[\mathfrak{X}(M)]^k \to \mathfrak{X}(M)$ if $k \in \mathbb{N}\setminus\{0\}$, and a vector field on M, if $k=0$. They constitute a $C^\infty(M)$-module, denoted by $\Psi^k(M)$. In particular, the elements of $\Psi^1(M)$ are just the $(1,1)$ tensor fields on M.

1.4. The *Frölicher - Nijenhuis bracket* of a vector 1-form $K \in \Psi^1(M)$ and a vector field $Y \in \mathfrak{X}(M)$ is the vector 1-form $[K,Y]$ defined by

$$[K,Y](X) = [K(X),Y] - K[X,Y], \quad X \in \mathfrak{X}(M). \tag{1.4a}$$

The Frölicher - Nijenhuis bracket of the vector 1-forms $K, L \in \Psi(M)$ is the vector 2-form $[K,L] \in \Psi^2(M)$, given by

$$\begin{aligned} [K,L](X,Y) = {} & [K(X),L(Y)] + [L(X),K(Y)] + K \circ L[X,Y] + \\ & + L \circ K[X,Y] - K[X,L(Y)] - K[L(X),Y] - \\ & - L[X,K(Y)] - L[K(X),Y]; \qquad X,Y \in \mathfrak{X}(M). \end{aligned} \tag{1.4b}$$

In particular,

$$\begin{aligned} \frac{1}{2}[K,K](X,Y) = {} & [K(X),K(Y)] + K^2[X,Y] - \\ & - K[X,K(Y)] - K[K(X),Y]. \end{aligned} \tag{1.4c}$$

$$N_K := \frac{1}{2}[K,K] \tag{1.4d}$$

is said to be the *Nijenhuis torsion* of K.

1.5. The *adjoint operator* $K^* : \Omega(M) \to \Omega(M)$, $\omega \mapsto K^*\omega$ of a vector 1-form $K \in \Psi^1(M)$ is defined by the value of $K^*\omega$ on k vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$ through the formula

$$K^*\omega(X_1, \dots, X_k) := \omega(K(X_1), \dots, K(X_k)) \tag{1.5}$$

if $k \neq 0$, and $\forall f \in C^\infty(M): \ K^*f := f$.

1.6. Any vector 1-form K determines two derivations of $\Omega(M)$. One of them denoted by $\imath_K$ is defined on the analogy of the substitution operator in the following way:

$$\forall f \in C^\infty(M) : \imath_K f = 0; \tag{1.6a}$$

$$\imath_K\omega(X_1, \dots, X_k) := \sum_{i=1}^{k} \omega(X_1, \dots, K(X_i), \dots, X_k) \tag{1.6b}$$

$$(\omega \in \Omega^k(M); \ X_i \in \mathfrak{X}(M), \ 1 \leq i \leq k).$$

$\imath_K$ is a derivation of degree 0. The other one denoted by d_K is the mapping

$$d_K := [\imath_K, d] := \imath_K \circ d - d \circ \imath_K; \tag{1.6c}$$

it is of degree 1. As an easy consequence of the definitions the following rule of anticommutativity holds:

$$d \circ d_K = -d_K \circ d. \tag{1.6d}$$

Evaluating d_K on a function $f \in C^\infty(M)$, we get

$$d_K f = \imath_K df = K^* df. \tag{1.6e}$$

Finally, we summarize some algebraical identities relating these operators to the Frölicher – Nijenhuis bracket (see [43]):

$$[fK, X] = f[K, X] - (Xf)K; \tag{1.6f}$$

$$[K, fL] = f[K, L] + d_K f \wedge L - df \wedge K \circ L; \tag{1.6g}$$

$$[X, \omega \otimes Y] = \mathcal{L}_X\omega \otimes Y + [X, Y] \otimes \omega; \tag{1.6h}$$

$$[K, \omega \otimes X] = d_K\omega \otimes X - d\omega \otimes K(X) + (-1)^k \omega \wedge [K, X] \tag{1.6i}$$

$$(K, L \in \Psi^1(M); \omega \in \Omega^k(M); \ X, Y \in \mathfrak{X}(M)).$$

1.7. $\pi : TM \to M$ is the *tangent bundle* of M; it is also denoted by τ_M. $\pi_0 : \mathcal{T}M \to M$ is the subbundle of τ_M constituted by the nonzero tangent vectors to M. The kernel of the tangent map $T\pi$ (or $T\pi_0$) is a canonical subbundle of τ_{TM} (or $\tau_{\mathcal{T}M}$), called the *vertical subbundle* and denoted by τ_{TM}^v (and $\tau_{\mathcal{T}M}^v$, resp.). The sections of the bundles τ_{TM}^v and $\tau_{\mathcal{T}M}^v$ are called *vertical vector fields*; the $C^\infty(TM)$-modules of vertical vector fields are denoted by $\mathfrak{X}^v(TM)$ and $\mathfrak{X}^v(\mathcal{T}M)$, respectively.

1.8. Tangent bundle geometry is dominated by two canonical objects: the *Liouville vector field* $C \in \mathfrak{X}^v(TM)$, and the *vertical endomorphism* $J \in \Psi^1(TM)$ (for the definitions see e.g. [22]). We shall frequently use the following properties:

$$\operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^v(TM); \tag{1.8a}$$

$$J^2 = 0; \tag{1.8b}$$

$$N_J := \frac{1}{2}[J, J] = 0; \tag{1.8c}$$

$$[J, C] = J. \tag{1.8d}$$

In particular the derivations $\imath_J$ and d_J of $\Omega(TM)$ are called the *vertical derivation* and the *vertical differentiation*, respectively. A straightforward calculation shows that

$$\imath_C \circ d_J + d_J \circ \imath_C = \imath_J. \tag{1.8e}$$

1.9. A differential form $\omega \in \Omega^k(TM)$ $(k \neq 0)$ is *semibasic*, if for each vector field X on TM, $\imath_{JX}\omega = 0$. Analogously, a vector k-form $K \in \Psi^k(TM)$ is said to be semibasic, if

$$J \circ K = 0 \quad \text{and} \quad \forall X \in \mathfrak{X}(TM):\ i_{JX}K = 0. \tag{1.9}$$

1.10. A *semispray* on the manifold M is a mapping

$$S : TM \to TTM,\ v \mapsto S(v) \in T_vTM,$$

satisfying the following conditions:

$$S \textit{ is smooth on } \mathcal{T}M; \tag{1.10a}$$

$$JS = C. \tag{1.10b}$$

The semispray S is called a *spray*, if

$$S \textit{ is of class } C^1 \textit{ on } TM \tag{1.10c}$$

and

$$[C, S] = S. \tag{1.10d}$$

1.11. The *vertical lift* ([22], [41]) of a function $f \in C^\infty(M)$ is $f^v := f \circ \pi \in C^\infty(TM)$, the complete lift of f is the function

$$f^c : TM \to \mathbb{R},\ v \mapsto f^c(v) := df(v) = v(f).$$

The following "homogeneity" properties are immediate consequences of the definitions:

$$Cf^v = 0, \quad Cf^c = f^c; \tag{1.11a}$$

$$Sf^v = f^c, \tag{1.11b}$$

where S is an arbitrary semispray on M.

1.12. Lemma. *A smooth function φ on TM or $\mathcal{T}M$ is a vertical lift if and only if $d_J\varphi = 0$.*

Proof. According to (1.6e) and (1.8a) we conclude that $d_J\varphi = 0$ if and only if for any vertical vector field $X \in \mathfrak{X}^v(TM) : X\varphi = 0$, i.e. φ can be written in the form $\varphi = f \circ \pi$. □

1.13. Vector fields on TM are determined by their action on $\{ f^c \mid f \in C^\infty(M) \}$ (see [42]). Thus for any vector field $X \in \mathfrak{X}(M)$ there exist vector fields X^v, $X^c \in \mathfrak{X}(TM)$ such that

$$\forall f \in C^\infty(M) : X^v f^c = (Xf) \circ \pi = (Xf)^v; \tag{1.13a}$$

$$\forall f \in C^\infty(M) : X^c f^c = (Xf)^c. \tag{1.13b}$$

X^v and X^c are called the *vertical* and the *complete lift* of X, respectively.

From the formulas (1.13a) and (1.13b) it can be easily seen that

$$\forall f \in C^\infty(M) : X^v f^v = 0; \tag{1.13c}$$

$$\forall f \in C^\infty(M) : X^c f^v = X^v f^c = (Xf)^v. \tag{1.13d}$$

1.14. Lemma. *A vertical vector field $Y \in \mathfrak{X}^v(TM)$, is a vertical lift if and only if for any vector field X on M, $[X^v, Y] = 0$.*

Proof. Suppose that Y is a vertical lift, i.e. $Y = Z^v (Z \in \mathfrak{X}(M))$. Then for any function f on M we get

$$[X^v, Y]f^c = [X^v, Z^v]f^c = X^v(Z^v f^c) - Z^v(X^v f^c) =$$
$$\overset{(1.13a)}{=} X^v(Zf)^v - Z^v(Xf)^v \overset{(1.13c)}{=} 0.$$

Conversely, the vanishing of the Lie-bracket $[X^v, Y]$ implies that for any function f on M

$$0 = [X^v, Y]f^c = X^v(Yf^c) - Y(X^v f^c) \overset{(1.13a)}{=} X^v(Yf^c) - Y(Xf)^v = X^v(Yf^c)$$

since Y is a vertical vector field. According to Lemma 1.12 it means that for any function f on M, Yf^c is a vertical lift (as a function) and, consequently, Y is a vertical lift (as a vector field; cf. (1.13a)). □

1.15. Lemma. *If S is a semispray on M, then*

$$\forall Z \in \mathfrak{X}(\mathcal{T}M) : J[JZ, S] = JZ. \tag{1.15}$$

For a proof, see [10], p. 295.

1.16. Lemma. *For each vector fields $X, Y \in \mathfrak{X}(M)$ we have*

$$[X^v, Y^v] = 0, \qquad [X^v, Y^c] = [X, Y]^v, \qquad [X^c, Y^c] = [X, Y]^c; \tag{1.16a}$$

$$[C, X^v] = -X^v, \qquad [C, X^c] = 0; \tag{1.16b}$$

$$JX^c = X^v, \qquad [J, X^c] = 0, \qquad [J, X^v] = 0. \tag{1.16c}$$

1.17. Lemma (1st local basis property). *If $(X_i)_{i=1}^n$ is a local basis for the module $\mathfrak{X}(M)$, then $(X_i^v, X_i^c)_{i=1}^n$ is a local basis for $\mathfrak{X}(TM)$.*

2. Horizontal endomorphisms and Finsler connections

2.1. In the sequel we shall introduce some important vector forms over TM. In conformity with the demands of Finsler geometry, their smoothness will *not* be required or assured *a priori* on the whole tangent manifold TM.

2.2. Definitions.
(i) A vector 1-form $h \in \Psi^1(TM)$, smooth only on $\mathcal{T}M$, is said to be a *horizontal endomorphism* on M, if it is a *projector* (i.e. $h^2 = h$) and $\operatorname{Ker} h = \mathfrak{X}^v(TM)$.
(ii) Assume $h \in \Psi^1(TM)$ is a horizontal endomorphism. The mapping

$$X \in \mathfrak{X}(M) \mapsto X^h := hX^c \in \mathfrak{X}(\mathcal{T}M) \tag{2.2a}$$

is called the *horizontal lifting* determined by h. The vector 1-form

$$H := [h, C] \in \Psi^1(TM) \tag{2.2b}$$

is said to be the *tension* of h. If $H = 0$, then h is called *homogeneous.* The vector 2-form

$$t := [J, h] \in \Psi^2(TM) \tag{2.2c}$$

and the vector 1-form

$$T := \imath_S t + H \tag{2.2d}$$

are said to be the *weak* and the *strong torsion* (S is an arbitrary semispray on M), respectively. The *curvature* of the horizontal endomorphism h is the vector 2-form

$$\Omega := -N_h = -\frac{1}{2}[h, h]. \tag{2.2e}$$

2.3. Remark. We emphasize again the condition of differentiability about a horizontal endomorphism is prescribed only on $\mathcal{T}M$. As a consequence, the smoothness of the tension, the torsion and the curvature is also guaranteed only over $\mathcal{T}M$.

2.4. Remark. The following relations are easy consequences of the definitions:

$$h \circ J = 0, \quad J \circ h = J. \tag{2.4a}$$

$$\forall X \in \mathfrak{X}(M): \ JX^h = X^v. \tag{2.4b}$$

$$\forall X, Y \in \mathfrak{X}(M): \ J[X^h, Y^h] = [X, Y]^v. \tag{2.4c}$$

2.5. Lemma and definition. *Suppose that h is a horizontal endomorphism on M.*
(a) *If $\mathfrak{X}^h(TM) := \operatorname{Im} h$, then $\mathfrak{X}(TM) = \mathfrak{X}^v(TM) \oplus \mathfrak{X}^h(TM)$ (direct sum). $\mathfrak{X}^h(TM)$ is called the module of* horizontal vector fields. *$v := 1_{\mathfrak{X}(TM)} - h$ is also a projector, the* vertical projection *on $\mathfrak{X}^v(TM)$ along $\mathfrak{X}^h(TM)$.*
(b) (2nd local basis property.) *If $(X_i)_{i=1}^n$ is a local basis of $\mathfrak{X}(M)$, then $\left(X_i^v, X_i^h\right)_{i=1}^n$ is a local basis of $\mathfrak{X}(TM)$.*

Proof. Trivial. ☐

2.6. Lemma. *The vector forms H, t, T and Ω are all semibasic, so they are completely determined by their action on completely (or horizontally) lifted vector fields. Namely, for any vector fields X, Y on M,*

$$H(X^c) = [X^h, C]; \tag{2.6a}$$

$$t(X^c, Y^c) = [X^h, Y^v] - [Y^h, X^v] - [X, Y]^v; \tag{2.6b}$$

$$T(X^c) = v[S, X^v] + X^h - X^c, \tag{2.6c}$$

where $S := h(S')$ (S' is an arbitrary semispray on M);

$$\Omega(X^c, Y^c) = -v[X^h, Y^h]. \tag{2.6d}$$

Proof. Straightforward calculations. □

2.7. Lemma. *If h is a horizontal endomorphism on M, then there exists a unique almost complex structure F ($F^2 = -1_{\mathfrak{X}(TM)}$) on TM, smooth over $\mathcal{T}M$, such that*

$$F \circ J = h, \quad F \circ h = -J. \tag{2.7a}$$

Explicitly

$$F = h[S, h] - J, \tag{2.7b}$$

where $S := h(S')$ (S' is an arbitrary semispray on M).

For a proof see [10], p. 314.

2.8. The following formulas can be obtained easily:

$$J \circ F = v, \quad F \circ v = h \circ F. \tag{2.8a}$$

$$v \circ F = F - F \circ v = F - h \circ F = -J. \tag{2.8b}$$

$$J[C, F] = v - [C, h]. \tag{2.8c}$$

2.9. Definitions. Suppose that h is a horizontal endomorphism on M and consider the almost complex structure F characterized by Lemma 2.7. Let, furthermore, D be a linear connection on the manifold TM or $\mathcal{T}M$. The pair (D, h) is said to be a *Finsler connection* on M if it satisfies the following conditions:

$$Dh = 0 \quad (D \text{ is reducible}); \tag{2.9a}$$

$$DF = 0 \quad (D \text{ is almost complex}). \tag{2.9b}$$

The mappings

$$D^h : (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto D^h_X Y := D_{hX} Y \in \mathfrak{X}(TM), \tag{2.9c}$$

$$D^v : (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto D^v_X Y := D_{vX} Y \in \mathfrak{X}(TM) \tag{2.9d}$$

are called the *h-covariant* and the *v-covariant differentiation* with respect to (D, h), respectively. The *h-deflection* of (D, h) is the mapping

$$h^*(DC) : X \in \mathfrak{X}(TM) \mapsto DC(hX) = D_{hX} C, \tag{2.9e}$$

while the *v-deflection* is $v^*(DC)$. The covariant differential DC is called the *deflection map.*

2.10. Remark. Assume that (D, h) is a Finsler connection on M. Applying (2.9a) we get immediately that

$$Y \in \mathfrak{X}^v(TM) \implies \forall X \in \mathfrak{X}(TM) : D_X Y \in \mathfrak{X}^v(TM), \tag{2.10a}$$

$$Y \in \mathfrak{X}^h(TM) \implies \forall X \in \mathfrak{X}(TM) : D_X Y \in \mathfrak{X}^h(TM). \tag{2.10b}$$

Owing to the condition (2.9b) it follows that D is completely determined by its action on $\mathfrak{X}(TM) \times \mathfrak{X}^v(TM)$. Namely, for each vector fields X, Y on TM,

$$D_{vX} hY = F D_{vX} JY, \tag{2.10c}$$

$$D_{hX} hY = F D_{hX} JY. \tag{2.10d}$$

Moreover, D is almost tangent as well:

$$DJ = 0. \tag{2.10e}$$

2.11. Lemma and definition. *Let (D, h) be a Finsler connection on M. Then the torsion tensor field $\mathbb{T}$ of D is completely determined by the following mappings:*

$$\mathbb{A}(X, Y) := h\mathbb{T}(hX, hY) - (h)h\text{-torsion}, \tag{2.11a}$$

$$\mathbb{B}(X, Y) := h\mathbb{T}(hX, vY) - (h)hv\text{-torsion}, \tag{2.11b}$$

$$\mathbb{R}^1(X, Y) := v\mathbb{T}(hX, hY) - (v)h\text{-torsion}, \tag{2.11c}$$

$$\mathbb{P}^1(X, Y) := v\mathbb{T}(hX, vY) - (v)hv\text{-torsion}, \tag{2.11d}$$

$$\mathbb{S}^1(X, Y) := v\mathbb{T}(vX, vY) - (v)v\text{-torsion}. \tag{2.11e}$$

2.12. Lemma and definition. *If (D, h) is a Finsler connection on M, then the curvature tensor field $\mathbb{K}$ of D is uniquely determined by the following three mappings:*

$$\mathbb{R}(X, Y)Z := \mathbb{K}(hX, hY)JZ - h\text{-curvature}, \tag{2.12a}$$

$$\mathbb{P}(X, Y)Z := \mathbb{K}(hX, JY)JZ - hv\text{-curvature}, \tag{2.12b}$$

$$\mathbb{Q}(X, Y)Z := \mathbb{K}(JX, JY)JZ - v\text{-curvature}. \tag{2.12c}$$

2.13. Example 1: horizontal lift of a linear connection. Suppose that ∇ is a *linear connection* on the manifold M. It is well-known that ∇ induces a homogeneous horizontal structure $h \in \Psi^1(TM)$, which is smooth on the whole tangent manifold TM. In this case

$$\forall X, Y \in \mathfrak{X}(M) : (\nabla_X Y)^v = [X^h, Y^v].$$

It is also known (see e.g. [22]), that there exists a unique linear connection $\overset{h}{\nabla}$ on the manifold TM, characterized by the following rules of calculation:

$$\begin{aligned} &\overset{h}{\nabla}_{X^v} Y^v = 0, \quad \overset{h}{\nabla}_{X^h} Y^v = (\nabla_X Y)^v = [X^h, Y^v], \\ &\overset{h}{\nabla}_{X^v} Y^h = 0, \quad \overset{h}{\nabla}_{X^h} Y^h = (\nabla_X Y)^h; \qquad X, Y \in \mathfrak{X}(M). \end{aligned} \tag{2.13}$$

$\overset{h}{\nabla}$ is called the *horizontal lift* of the linear connection ∇. Now it is easy to check that $(\overset{h}{\nabla}, h)$ satisfies the conditions (2.9a), (2.9b); therefore $(\overset{h}{\nabla}, h)$ is a Finsler-connection on M.

2.14. Example 2: Berwald-type connections. Let a horizontal endomorphism h on the manifold M be given. Define the mapping

$$\overset{\circ}{D} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M),\ (X,Y) \mapsto \overset{\circ}{D}_X Y$$

as follows:

$$\overset{\circ}{D}_{JX} JY := J[JX, Y], \tag{2.14a}$$

$$\overset{\circ}{D}_{hX} JY := v[hX, JY], \tag{2.14b}$$

$$\overset{\circ}{D}_{JX} hY := h[JX, Y], \tag{2.14c}$$

$$\overset{\circ}{D}_{hX} hY := hF[hX, JY] \tag{2.14d}$$

and

$$\overset{\circ}{D}_X Y := \overset{\circ}{D}_{vX} vY + \overset{\circ}{D}_{hX} vY + \overset{\circ}{D}_{vX} hY + \overset{\circ}{D}_{hX} hY.$$

Then $\overset{\circ}{D}$ is obviously a linear connection on $\mathcal{T}M$. It is easy to check that $(\overset{\circ}{D}, h)$ satisfies (2.9a) and (2.9b).

2.15. Proposition. *Suppose that $(\overset{\circ}{D}, h)$ is a Finsler connection of Berwald-type on M. Then the h-curvature $\overset{\circ}{\mathbb{R}}$, the hv-curvature $\overset{\circ}{\mathbb{P}}$ and the v-curvature $\overset{\circ}{\mathbb{Q}}$ of $\overset{\circ}{D}$ are semibasic and*

$$\forall X, Y, Z \in \mathfrak{X}(M) : \overset{\circ}{\mathbb{R}}(X^c, Y^c)Z^c = [J, \Omega(X^c, Y^c)]Z^h; \tag{2.15a}$$

$$\forall X, Y, Z \in \mathfrak{X}(M) : \overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c = \overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h = \big[[X^h, Y^v], Z^v\big]; \tag{2.15b}$$

$$\overset{\circ}{\mathbb{Q}} = 0. \tag{2.15c}$$

Proof. It is obvious from (2.12a)–(2.12c) that $\overset{\circ}{\mathbb{R}}$, $\overset{\circ}{\mathbb{P}}$ and $\overset{\circ}{\mathbb{Q}}$ are indeed semibasic. Hence, in view of Lemma 1.17 and Lemma 2.5(b), they are completely determined by their action on the triplets of form

$$(X^c, Y^c, Z^c) \quad \text{or} \quad (X^h, Y^h, Z^h); \qquad X, Y, Z \in \mathfrak{X}(M).$$

(a) Taking into account that for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$[X^h, Y^v],\ \big[X^h, [Y^h, Z^v]\big],\ \dots \text{ are vertical; } h[X^h, Y^h] = [X, Y]^h,$$

$$JX^h = JX^c = X^v \text{ (see (2.4b) and (1.16c)), } J \circ F = v \text{ (from (2.8a));}$$

and applying the rules of calculation (2.14a), (2.14b) and the Jacobi identity for the Lie bracket of vector fields, we obtain:

$$\begin{aligned}\mathring{\mathbb{R}}(X^c, Y^c)Z^c := \mathring{\mathbb{K}}(hX^c, hY^c)JZ^c = \mathring{\mathbb{K}}(X^h, Y^h)Z^v = \mathring{D}_{X^h}\mathring{D}_{Y^h}Z^v - \\ - \mathring{D}_{Y^h}\mathring{D}_{X^h}Z^v - \mathring{D}_{[X^h,Y^h]}Z^v = \mathring{D}_{X^h}[Y^h, Z^v] - \mathring{D}_{Y^h}[X^h, Z^v] - \\ - \mathring{D}_{[X,Y]^h}Z^v - \mathring{D}_{v[X^h,Y^h]}Z^v = \\ = \big[X^h, [Y^h, Z^v]\big] - \big[Y^h, [X^h, Z^v]\big] - \big[[X,Y]^h, Z^v\big] - \\ - J\big[v[X^h, Y^h], Z^h\big] = \\ = -\big[Z^v, [X,Y]^h\big] - \big[Z^v, v[X^h, Y^h]\big] + \big[Z^v, [X,Y]^h\big] - \\ - J\big[v[X^h, Y^h], Z^h\big] = -\big[JZ^h, v[X^h, Y^h]\big] + J\big[Z^h, v[X^h, Y^h]\big] = \\ = -\big[J, v[X^h, Y^h]\big]Z^h \overset{(2.6d)}{=} [J, \Omega(X^c, Y^c)]Z^h.\end{aligned}$$

Since $\mathring{\mathbb{R}}$ is semibasic, this proves (2.15a).

(b) Computing and arguing as before, we find that

$$\begin{aligned}\mathring{\mathbb{P}}(X^c, Y^c)Z^c := \mathring{\mathbb{K}}(hX^c, JY^c)JZ^c = \\ = -J\big[Y^v, F[X^h, Z^v]\big] - J\big[[X^h, Y^v], Z^c\big].\end{aligned}$$

Since by (1.16c) $[J, Z^c] = 0$, we can write

$$\begin{aligned}0 = [J, Z^c][X^h, Y^v] = \big[J[X^h, Y^v], Z^c\big] - J\big[[X^h, Y^v], Z^c\big] = \\ = -J\big[[X^h, Y^v], Z^c\big],\end{aligned}$$

and so

$$\mathring{\mathbb{P}}(X^c, Y^c)Z^c = J\big[F[X^h, Z^v], Y^v\big].$$

Now applying (1.16c) again, we obtain that

$$0 = [J, Y^v]F[X^h, Z^v] = \big[JF[X^h, Z^v], Y^v\big] - J\big[F[X^h, Z^v], Y^v\big],$$

and thus

$$\begin{aligned}J\big[F[X^h, Z^v], Y^v\big] \overset{(2.8a)}{=} \big[[X^h, Z^v], Y^v\big] = -\big[[Z^v, Y^v], X^h\big] - \\ - \big[[Y^v, X^h], Z^v\big] \overset{(1.16a)}{=} \big[[X^h, Y^v], Z^v\big];\end{aligned}$$

hence our assertion.

(c) A quick and easy calculation yields the relation (2.15c) and we omit it. □

2.16. We recall that if h is a horizontal endomorphism on M,then there always exists a semispray on M, which is horizontal with respect to h. To see this, consider an arbitrary semispray S' on M. If $S = hS'$, then $JS = J(hS') \overset{(2.4a)}{=} JS' = C$ and hence S is a semispray. Since $hS = h^2S' = hS' = S$, S is horizontal with respect to h. It is also clear that S does not depend on the choice of S'.

The spray S constructed in this way is called *the semispray associated with* h (cf. [10], p. 306).

2.17. Proposition. *Suppose that h is a homogeneous horizontal endomorphism on M and let $(\overset{\circ}{D}, h)$ be the Berwald-type Finsler connection induced by h. If S is a semispray on M, then*

$$\forall X, Y \in \mathfrak{X}(\mathcal{T}M) : \overset{\circ}{\mathbb{R}}(X, Y)S = \Omega(X, Y). \tag{2.17}$$

Proof. Since $\overset{\circ}{\mathbb{R}}$ and Ω are semibasic, it is enough to check that

$$\forall X, Y \in \mathfrak{X}(M) : \overset{\circ}{\mathbb{R}}(X^h, Y^h)S = \Omega(X^h, Y^h). \tag{2.17a}$$

We can also assume that S is the semispray associated with h. Then $hS = S$ and (2.15a) yields the relation

$$\begin{aligned}
\overset{\circ}{\mathbb{R}}(X^h, Y^h)S = [J, \Omega(X^h, Y^h)]S &= [C, \Omega(X^h, Y^h)] - J[S, \Omega(X^h, Y^h)] \\
&\overset{(2.6d),(2.8a)}{=} \big[v[X^h, Y^h], C\big] + J\big[JF\Omega(X^h, Y^h), S\big] \\
&\overset{\text{Lemma 1.15}}{=} \big[v[X^h, Y^h], C\big] + \Omega(X^h, Y^h).
\end{aligned}$$

It remains only to show that the first term on the right hand side vanishes. In view of the homogeneity,

$$[v, C] = 0 \quad \text{and} \quad [X^h, C] = 0,$$

thus, in particular,

$$0 = [v, C][X^h, Y^h] = \big[v[X^h, Y^h], C\big] - v\big[[X^h, Y^h], C\big].$$

Finally, using the Jacobi identity and the homogeneity of h over again, we obtain that

$$0 = \big[[X^h, Y^h], C\big] + \big[[Y^h, C], X^h\big] + \big[[C, X^h], Y^h\big] = \big[[X^h, Y^h], C\big]$$

which completes the proof. □

2.18. Corollary. *Hypothesis as in Proposition 2.17. Then*

$$\overset{\circ}{\mathbb{R}} = 0 \iff \Omega = 0.$$

Proof. If Ω vanishes, then $\overset{\circ}{\mathbb{R}}$ also vanishes by (2.15a). Conversely, it follows from (2.17) that $\overset{\circ}{\mathbb{R}} = 0 \implies \Omega = 0$. □

3. Finsler manifolds

3.1. Definitions. Let a function $E : TM \to \mathbb{R}$ be given. The pair (M, E), or simply M, is said to be a *Finsler manifold*, if the following conditions are satisfied:

(3.1a) $\forall a \in \mathcal{T}M : E(a) > 0;\ E(0) = 0.$

(3.1b) E is of class C^1 on TM and smooth over $\mathcal{T}M$.

(3.1c) $CE = 2E$; i.e. E is homogeneous of degree 2.

(3.1d) The *fundamental form* $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$ is nondegenerate.

The function E is called the *energy function* of the Finsler manifold. A horizontal endomorphism on M is said to be *conservative* if $d_h E = 0$.

3.2. Metrics. Assume (M, E) is a Finsler manifold with fundamental form ω.

(a) The mapping

(3.2a) $$g : \mathfrak{X}^v(\mathcal{T}M) \times \mathfrak{X}^v(\mathcal{T}M) \to C^\infty(\mathcal{T}M), (JX, JY) \mapsto g(JX, JY) := \omega(JX, Y)$$

is a well-defined, nondegenerate, symmetric bilinear form which is said to be the *Riemann–Finsler metric* of (M, E). The Finsler manifold is called *positive definite* if g is positive definite.

(b) Suppose that h is a horizontal endomorphism on M, $v = 1_{\mathfrak{X}(\mathcal{T}M)} - h$. Then

(3.2b) $$\begin{gathered} g_h : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to C^\infty(\mathcal{T}M), \\ (X, Y) \mapsto g_h(X, Y) := g(JX, JY) + g(vX, vY) \end{gathered}$$

is a pseudo-Riemannian metric on $\mathcal{T}M$, called the *prolongation of* g along h.

(c) It follows at once from (3.2b) that

(3.2c) $$\forall X, Y \in \mathfrak{X}(\mathcal{T}M) : g_h(hX, JY) = 0.$$

3.3. The following formulas can be easily obtained:

(3.3a) $g(C, C) = g_h(C, C) = 2E;$

(3.3b) $\forall X, Y \in \mathfrak{X}(M) : g(X^v, Y^v) = g_h(X^v, Y^v) = X^v(Y^v E);$

(3.3c) $\imath_J \omega = 0, \quad \imath_C \omega = d_J E;$

(3.3d) $\mathcal{L}_C \omega = \omega, \quad \mathcal{L}_C d_J E = d_J E,$

where h is an arbitrary horizontal endomorphism on the Finsler manifold (M, E).

3.4. The Cartan tensors. Let a Finsler manifold (M, E) be given. Suppose that h is a horizontal endomorphism on M and let g_h be the prolongation of g along h.

(i) There exists a unique tensor

$$\mathcal{C} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$$

such that $J \circ \mathcal{C} = 0$ and

$$\text{(3.4a)} \qquad \forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : g_h(\mathcal{C}(X,Y), JZ) = \frac{1}{2}\left(\mathcal{L}_{JX} J^* g_h\right)(Y, Z).$$

The tensor $\mathcal{C}$, as well as its lowered tensor $\mathcal{C}_\flat$ defined by

$$\text{(3.4b)} \quad \forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : \mathcal{C}_\flat(X,Y,Z) := g_h(\mathcal{C}(X,Y), JZ) \overset{(3.3b)}{=} g(\mathcal{C}(X,Y), JZ)$$

is called the *first Cartan tensor* of the Finsler manifold.

(ii) Analogously, we introduce a tensor

$$\mathcal{C}' : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$$

by the conditions $J \circ \mathcal{C}' = 0$ and

$$\text{(3.4c)} \qquad \forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : g_h(\mathcal{C}'(X,Y), JZ) = \frac{1}{2}\left(\mathcal{L}_{hX} g_h\right)(JY, JZ).$$

Then $\mathcal{C}'$ is well-defined; it is called the *second Cartan tensor* of the Finsler manifold, belonging to the horizontal endomorphism h. We use the same terminology also for the lowered tensor $\mathcal{C}'_\flat$.

3.5. Remark. $\mathcal{C}$ and $\mathcal{C}'$ are clearly semibasic. We shall see soon that $\mathcal{C}$ is independent of the choice of the horizontal endomorphism h, it depends on the energy function alone (i.e., on the Finsler structure). On the other hand, the second Cartan tensor $\mathcal{C}'$ strongly depends on the horizontal endomorphism!

3.6. Lemma. *Let (M, E) be a Finsler manifold and $(\overset{\circ}{D}, h)$ a Berwald-type Finsler connection on M. Then for each vector fields X, Y, Z on M,*

$$\text{(3.6a)} \qquad 2\mathcal{C}_\flat(X^c, Y^c, Z^c) = \left(\overset{\circ}{D}_{X^v} g_h\right)(Y^v, Z^v) = X^v[Y^v(Z^v E)];$$

$$\text{(3.6b)} \qquad 2\mathcal{C}'_\flat(X^c, Y^c, Z^c) = \left(\overset{\circ}{D}_{X^h} g_h\right)(Y^v, Z^v) = \left[Y^v, [X^h, Z^v]\right]E + \\ + Y^v[Z^v(X^h E)]$$

($\mathcal{C}'$ is the second Cartan tensor belonging to h).

Proof. Straightforward calculations. □

3.7. Corollary. *The first Cartan tensor is symmetric, the lowered first Cartan tensor is totally symmetric.*

Proof. We infer this immediately from (3.6a): since the Lie bracket of vertically lifted vector fields vanishes, $X^v[Y^v(Z^v E)]$ does not depend on the order of the vector fields. □

3.8. Lemma. *If S is an arbitrary semispray, then $\imath_S\mathcal{C} = \imath_S\mathcal{C}_\flat = 0$.*

Proof. Due to symmetry, it is enough to check that for each vector fields X,Y on M we have $\mathcal{C}_\flat(S, X, Y) = 0$. From (3.6a)

$$2\mathcal{C}_\flat(S, X^c, Y^c) = \left(\overset{\circ}{D}_{JS}g_h\right)(X^v, Y^v) = Cg_h(X^v, Y^v) - g_h(\overset{\circ}{D}_C X^v, Y^v) -$$

$$- g_h(X^v, \overset{\circ}{D}_C Y^v) = C[X^v(Y^v E)],$$

since e.g.

$$\overset{\circ}{D}_C X^v = \overset{\circ}{D}_{JS} JX^c \overset{(2.14a)}{=} J[C, X^c] \overset{(1.16b)}{=} 0.$$

Using systematically (1.16b) and (3.1c) we obtain the vanishing of $C[X^v(Y^v E)]$; hence our assertion. □

3.9. Proposition. *Let (M, E) be a Finsler manifold. If h is a conservative, torsion-free horizontal endomorphism then the lowered second Cartan tensor is totally symmetric.*

Proof. Since h is conservative, for each vector field X on M,

$$0 = (d_h E)(X^c) \overset{(1.6e)}{=} (i_h dE)(X^c) = (dE)X^h = X^h E,$$

so it follows from (3.6b) that

$$\forall X, Y, Z \in \mathfrak{X}(M) : 2\mathcal{C}'_\flat(X^c, Y^c, Z^c) = \big[Y^v, [X^h, Z^v]\big]E.$$

Now, using the Jacobi identity and the condition $t = 0$, we obtain

$$\begin{aligned} 0 &= \big[Y^v, [X^h, Z^v]\big] + \big[X^h, [Z^v, Y^v]\big] + \big[Z^v, [Y^v, X^h]\big] \overset{(1.16a)}{=} \\ &= \big[Y^v, [X^h, Z^v]\big] + \big[Z^v, [Y^v, X^h]\big] \overset{(2.6b)}{=} \big[Y^v, [X^h, Z^v]\big] + \\ &\quad + \big[Z^v, -[Y^h, X^v]\big] + \big[Z^v, -[X, Y]^v\big] = \big[Y^v, [X^h, Z^v]\big] - \big[Z^v, [Y^h, X^v]\big]; \end{aligned}$$

therefore $\big[Y^v, [X^h, Z^v]\big] = \big[Z^v, [Y^h, X^v]\big]$. This means that

$$\mathcal{C}'_\flat(X^c, Y^c, Z^c) = \mathcal{C}'_\flat(Y^c, Z^c, X^c).$$

The other symmetries of $\mathcal{C}'_\flat$ can be shown in the same manner. □

3.10. Example: Randers manifolds. Let α be a Riemannian metric and β a nonzero 1-form on the manifold M. Consider the functions

$$\begin{aligned} &L_\alpha : TM \to \mathbb{R}, && v \to L_\alpha(v) := [\alpha_{\pi(v)}(v, v)]^{1/2}; \\ &\widetilde{\beta} : TM \to \mathbb{R}, && v \to \widetilde{\beta}(v) := \beta_{\pi(v)}(v); \\ &L := L_\alpha + \widetilde{\beta}; && E := \frac{1}{2}L^2. \end{aligned} \tag{3.10}$$

If

$$\|\widetilde{\beta}\| := \sup_{v \in \mathcal{T}M} \frac{\widetilde{\beta}(v)}{L_\alpha(v)} < 1,$$

then (M, E) is a Finsler manifold which is said to be the *Randers manifold* constructed from the Riemann manifold (M, α) by perturbation with $\widetilde{\beta}$.

(a) The Riemann– Finsler metric of a Randers manifold can be represented in the form:

$$g = \frac{L}{L_\alpha}\overline{\alpha} - \frac{\widetilde{\beta}}{L_\alpha^3}\imath_C\overline{\alpha} \otimes \imath_C\overline{\alpha} + \frac{1}{L_\alpha}\imath_C\overline{\alpha} \otimes d_v\widetilde{\beta} + \\ + \frac{1}{L_\alpha}d_v\widetilde{\beta} \otimes \imath_C\overline{\alpha} + d_v\widetilde{\beta} \otimes d_v\widetilde{\beta}, \tag{3.10a}$$

where $v := 1_{\mathfrak{X}(TM)} - h$ and for any vector fields $X, Y \in \mathfrak{X}(M)$,

$$\overline{\alpha}(X^v, Y^v) = [\alpha(X, Y)]^v.$$

(b) The lowered first Cartan tensor of a Randers manifold can be given by the formula

$$\mathcal{C}_\flat = \frac{3}{2}\Bigg[\frac{\widetilde{\beta}}{L_\alpha^5}J^*\imath_C\overline{\alpha} \otimes J^*\imath_C\overline{\alpha} \otimes J^*\imath_C\overline{\alpha} + \\ + \operatorname{Sym}\Big(\frac{1}{L_\alpha}J^*\overline{\alpha} \otimes d_J\widetilde{\beta} - \frac{\widetilde{\beta}}{L_\alpha^3}J^*\overline{\alpha} \otimes J^*\imath_C\overline{\alpha} - \frac{1}{L_\alpha^3}J^*\imath_C\overline{\alpha} \otimes J^*\imath_C\overline{\alpha} \otimes d_J\widetilde{\beta}\Big)\Bigg]. \tag{3.10b}$$

For the formulas (3.10a) and (3.10b) see [38].

(c) The Japanese school of differential geometry has played a dominant role in development of the theory of Finsler manifolds with so-called (α, β)-metrics (especially Randers manifolds); lots of interesting historical remarks and further facts can be found in Matsumoto's synthetical report [26] (see also [15], [17] and [23]).

4. Notable Finsler connections

4.1. Lemma and definition. *On any Finsler manifold* (M, E) *there is a spray* $S : TM \to TTM$*, uniquely determined on* $\mathcal{T}M$ *by the relation*

$$\imath_S \omega = -dE \tag{4.1}$$

and prolonged to a C^1*-mapping of* TM *such that* $S \restriction TM \setminus \mathcal{T}M = 0$*. The spray* S *is called the canonical spray of the Finsler manifold.*

For a proof, see [10], p. 323 and [7], p. 60.

4.2. Theorem (Fundamental lemma of Finsler geometry)**.** *Let* (M, E) *be a Finsler manifold. There exists a unique horizontal endomorphism* h *on* M*,called the Barthel endomorphism, such that*

(a) $\qquad$ *h is conservative (i.e.,* $d_h E = 0$*),*

(b) $\qquad$ *h is homogeneous (i.e.,* $H = [h, C] = 0$*),*

(c) $\qquad$ *h is torsion-free (i.e.,* $t = [J, h] = 0$*).*

Explicitly,

$$h = \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, S]\right),$$

where S is the canonical spray.

The result is due to J. Grifone [10].

4.3. Theorem. *Let* (M, E) *be a Finsler manifold and let h be a horizontal endomorphism on* M*. There exists a unique Finsler connection* $(\overset{\circ}{D}, h)$ *on* M *such that*

(4.3a) $\qquad$ *the* $(v)hv$*-torsion* $\overset{\circ}{\mathbb{P}}{}^1$ *of* $\overset{\circ}{D}$ *vanishes*;

(4.3b) $\qquad$ *the* $(h)hv$*-torsion* $\overset{\circ}{\mathbb{B}}$ *of* $\overset{\circ}{D}$ *vanishes.*

This Finsler connection is of Berwald-type, so the covariant derivatives with respect to $\overset{\circ}{D}$ *can be calculated by* (2.14a)–(2.14d)*. If* $(\overset{\circ}{D}, h)$ *satisfies the further conditions*

(4.3c) $\qquad$ *h is conservative*;

(4.3d) $\qquad$ *the h-deflection* $h^*\overset{\circ}{D}C$ *vanishes,*

(4.3e) $\qquad$ *the* $(h)h$*-torsion* $\overset{\circ}{\mathbb{A}}$ *of* $\overset{\circ}{D}$ *vanishes,*

then h is just the Barthel endomorphism of the Finsler manifold.

4.4. Remark. The Finsler connection determined by (4.3a)–(4.3e) is said to be the *Berwald connection* of the Finsler manifold. The axioms presented here were at first formulated by T. OKADA [29], with a slight difference. The novelty of our approach consists in drawing a distinction between the roles of the first group (4.3a)–(4.3b) and of the second group (4.3c)–(4.3e) of axioms. For an intrinsic proof of the theorem, see [33].

4.5. Theorem and definition. *Let (M, E) be a Finsler manifold and suppose that h is a conservative torsion-free horizontal endomorphism on M. Let g_h be the prolongation of g along h and $\mathcal{C}'$ the second Cartan tensor belonging to h. There exists a unique Finsler connection (D, h) on M such that*

(4.5a) *D is metrical (i.e. $Dg_h = 0$);*

(4.5b) *the $(v)v$-torsion $\mathbb{S}^1$ of D vanishes;*

(4.5c) *the $(h)h$-torsion $\mathbb{A}$ of D vanishes.*

The covariant derivatives with respect to D can be explicitly calculated by the following formulas: for each vector fields X, Y on $\mathcal{T}M$,

$$D_{JX}JY = J[JX, Y] + \mathcal{C}(X, Y) = \overset{\circ}{D}_{JX}JY + \mathcal{C}(X, Y); \tag{4.5d}$$

$$D_{hX}JY = v[hX, JY] + \mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X, Y); \tag{4.5e}$$

$$D_{JX}hY = h[JX, Y] + F\mathcal{C}(X, Y) = \overset{\circ}{D}_{JX}hY + F\mathcal{C}(X, Y); \tag{4.5f}$$

$$D_{hX}hY = hF[hX, JY] + F\mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX}hY + F\mathcal{C}'(X, Y). \tag{4.5g}$$

Then

$$h^*DC = \frac{1}{2}H$$

where H is the tension of h (2.2b). *Therefore, if in addition to* (4.5a)–(4.5c)

$$h^*DC = 0 \tag{4.5h}$$

is also satisfied, then h is the Barthel endomorphism of the Finsler manifold. In this case (D, h) is called the (classical) Cartan connection *of the Finsler manifold (M, E).*

Proof. The idea of the *existence proof* is immediate. We start from a conservative, torsion-free horizontal endomorphism h (whose existence is clearly guaranteed; see also 4.6. Remarks (c)) and build the second Cartan tensor $\mathcal{C}'$ belonging to h. Then we define a rule of covariant differentation by the formulas (4.5d)–(4.5g). It can be checked by a straightforward calculation that the pair (D, h) obtained in this way is indeed a Finsler connection, and the axioms (4.5a)–(4.5c) are satisfied.

In our subsequent considerations we are going to prove the *unicity* statement. Assume (D, h) is a Finsler connection on M, satisfying (4.5a)–(4.5c). We show that the rules of calculation (4.5d)–(4.5g) are valid.

<u>1st step.</u> Applying the "Christoffel process", we derive (4.5d). We can restrict ourselves to vertically lifted vector fields. From condition (4.5a), for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned} X^v g_h(Y^v, Z^v) &= g_h(D_{X^v} Y^v, Z^v) + g_h(Y^v, D_{X^v} Z^v), \\ Y^v g_h(Z^v, X^v) &= g_h(D_{Y^v} Z^v, X^v) + g_h(Z^v, D_{Y^v} X^v), \\ -Z^v g_h(X^v, Y^v) &= -g_h(D_{Z^v} X^v, Y^v) - g_h(X^v, D_{Z^v} Y^v). \end{aligned}$$

Since from (4.5b) $D_{X^v} Y^v - D_{Y^v} X^v = [X^v, Y^v] = 0$ and so on, adding the above three equations we get the relation

$$\begin{aligned} g_h(2D_{X^v} Y^v, Z^v) &= X^v g_h(Y^v, Z^v) + Y^v g_h(Z^v, X^v) - Z^v g_h(X^v, Y^v) = \\ &\overset{(3.3b),(3.6a),3.7}{=} 2\mathcal{C}_\flat(X^c, Y^c, Z^c) = 2g_h(\mathcal{C}(X^c, Y^c), Z^v). \end{aligned}$$

Hence

$$D_{X^v} Y^v = \mathcal{C}(X^c, Y^c) = \mathcal{C}(hX^c + vX^c, hY^c + vY^c) = \mathcal{C}(X^h, Y^h),$$

from which (4.5d) easily follows. In view of (2.10c) this implies (4.5f).

<u>2nd step.</u> Now we apply the Christoffel process to the h-covariant derivatives of g_h. Then (4.5a) yields the relations

$$\text{(a)} \qquad \begin{cases} X^h g_h(Y^v, Z^v) = g_h(D_{X^h} Y^v, Z^v) + g_h(Y^v, D_{X^h} Z^v), \\ Y^h g_h(Z^v, X^v) = g_h(D_{Y^h} Z^v, X^v) + g_h(Z^v, D_{Y^h} X^v), \\ -Z^h g_h(X^v, Y^v) = -g_h(D_{Z^h} X^v, Y^v) - g_h(X^v, D_{Z^h} Y^v) \end{cases}$$

$(X, Y, Z \in \mathfrak{X}(M))$. From condition (4.5c), i.e., from the vanishing of the $(h)h$-torsion we conclude that e.g.

$$D_{X^h} Y^h - D_{Y^h} X^h = h[X^h, Y^h] = [X, Y]^h = h[X, Y]^c,$$

hence

$$FD_{X^h} Y^h - FD_{Y^h} X^h = Fh[X, Y]^c \overset{(2.7a),(1.16c)}{=} -[X, Y]^v.$$

So, taking into account (2.10d), we obtain the relations

$$\text{(b)} \qquad \begin{cases} D_{X^h} Y^v - D_{Y^h} X^v = [X, Y]^v, \\ D_{Y^h} Z^v - D_{Z^h} Y^v = [Y, Z]^v, \\ -D_{Z^h} X^v + D_{X^h} Z^v = -[Z, X]^v. \end{cases}$$

Adding now both sides of (a) and using (b), it follows that

$$\text{(c)} \qquad \begin{aligned} g_h(2D_{X^h} Y^v, Z^v) &= X^h g_h(Y^v, Z^v) + Y^h g_h(Z^v, X^v) - Z^h g_h(X^v, Y^v) + \\ &+ g_h([X, Y]^v, Z^v) - g_h([Y, Z]^v, X^v) + g_h([Z, X]^v, Y^v). \end{aligned}$$

3rd step. We apply the Christoffel process to the tensor $\mathcal{C}'$ belonging to h. For any vector fields $X, Y, Z \in \mathfrak{X}(M)$, we get:

$$2g_h(\mathcal{C}'(X^h, Y^h), Z^v) = X^h g_h(Y^v, Z^v) - g_h([X^h, Y^v], Z^v) - g_h(Y^v, [X^h, Z^v]),$$

$$2g_h(\mathcal{C}'(Y^h, Z^h), X^v) = Y^h g_h(Z^v, X^v) - g_h([Y^h, Z^v], X^v) - g_h(Z^v, [Y^h, X^v]),$$

$$-2g_h(\mathcal{C}'(Z^h, X^h), Y^v) = -Z^h g_h(X^v, Y^v) + g_h([Z^h, X^v], Y^v) + g_h(X^v, [Z^h, Y^v]).$$

Adding these three equations, in view of the symmetry of $\mathcal{C}'$ (assured by Proposition 3.9) we obtain:

$$\begin{aligned} g_h(2\mathcal{C}'(X^h, Y^h), Z^v) &= X^h g_h(Y^v, Z^v) + Y^h g_h(Z^v, X^v) - Z^h g_h(X^v, Y^v) - \\ &- g_h([X^h, Y^v] + [Y^h, X^v], Z^v) + g_h([Z^h, X^v] - [X^h, Z^v], Y^v) + \\ &+ g_h([Z^h, Y^v] - [Y^h, Z^v], X^v). \end{aligned} \tag{d}$$

From (c) and (d) it follows that

$$\begin{cases} g_h(2D_{X^h} Y^v, Z^v) = g_h(2\mathcal{C}'(X^h, Y^h), Z^v) + \\ \qquad + g_h([X^h, Y^v] + [Y^h, X^v] + [X, Y]^v, Z^v) + \\ \qquad + g_h([X^h, Z^v] - [Z^h, X^v] - [X, Z]^v, Y^v) + \\ \qquad + g_h([Y^h, Z^v] - [Z^h, Y^v] - [Y, Z]^v, X^v). \end{cases} \tag{e}$$

Since h is torsion-free, the last two terms on the right hand side of (e) vanish (cf. (2.6b)), while in the second term

$$[X^h, Y^v] + [Y^h, X^v] + [X, Y]^v = 2[X^h, Y^v].$$

Hence

$$g_h(2D_{X^h} Y^v, Z^v) = g_h(2\mathcal{C}'(X^h, Y^h) + 2[X^h, Y^v], Z^v),$$

which yields the formula

$$D_{X^h} Y^v = [X^h, Y^v] + \mathcal{C}'(X^h, Y^h).$$

This proves (4.5e) and, by (2.10d), (4.5g). We have thus established the unicity assertion.

4th step. We show that

$$h^* DC = \frac{1}{2} H.$$

Let X and Y be arbitrary vector fields on M. Then $h^* DC(X^h) = DC(X^h) = D_{X^h} C$ and

$$\begin{aligned} g_h(D_{X^h} C, Y^v) &= g_h(D_{X^h} JS, Y^v) \overset{(4.5e)}{=} g_h\big([X^h, C], Y^v\big) + \\ &+ g_h(\mathcal{C}'(X^h, S), Y^v) \overset{(3.4c)}{=} g_h\big([X^h, C], Y^v\big) + \frac{1}{2} X^h g_h(C, Y^v) - \\ &- \frac{1}{2} g_h\big([X^h, C], Y^v\big) - \frac{1}{2} g_h\big([X^h, Y^v], C\big) = \frac{1}{2} g_h\big([X^h, C], Y^v\big) + \\ &+ \frac{1}{2}\left(X^h(Y^v E) - [X^h, Y^v] E\right) = \frac{1}{2} g_h\big([X^h, C], Y^v\big) + \frac{1}{2} Y^v(X^h E) = \\ &\overset{d_h E = 0}{=} g_h(\frac{1}{2}[X^h, C], Y^v), \end{aligned}$$

from which follows that
$$\frac{1}{2}[X^h, C] = D_{X^h}C.$$
This means that $\frac{1}{2}H(X^h) = h^*DC(X^h)$, proving our assertion. $\square$

4.6. Remarks.
(a) Observe that axioms (4.5a) and (4.5h) imply for any Finsler connection (D, h) that h is a conservative horizontal endomorphism. Indeed, for each vector field X on M,

$$0 \overset{(4.5\text{a})}{=} (D_{X^h}g_h)(C, C) = X^h g_h(C, C) - 2g_h(D_{X^h}C, C) =$$
$$\overset{(3.3\text{a})}{=} 2X^h E - 2g_h(D_{X^h}C, C) \overset{(4.5\text{h})}{=} 2X^h E = 2(d_h E)(X^h),$$

which means that $d_h E = 0$.
(b) Axioms to characterize the classical Cartan connection were first formulated by M. Matsumoto; for an instructive historical remark see [24], p. 112.
(c) For the sake of completness we sketch an original process to construct torsion-free, conservative horizontal endomorphisms on a Finsler manifold (M, E). Let a function $\beta \in C^\infty(M)$ be given and define a semispray $\widetilde{S}$ by the formula

$$\widetilde{S} = S - \operatorname{grad}\beta^v, \tag{4.6}$$

where S denotes the canonical spray and $\operatorname{grad}\beta^v$ is the gradient of the function $\beta^v := \beta \circ \pi$ (see II.1.1). Then *the horizontal endomophism $\widetilde{h}$ induced by $\widetilde{S}$ is torsion-free and conservative*. Indeed, from the definition

$$\widetilde{h} := \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, \widetilde{S}]\right) = h - \frac{1}{2}[J, \operatorname{grad}\beta^v]$$

we get immediately that for any vector fields $X, Y \in \mathfrak{X}(M)$,

$$\widetilde{t}(X^c, Y^c) \overset{(2.6\text{b})}{=} [X^{\widetilde{h}}, Y^v] - [Y^{\widetilde{h}}, X^v] - [X, Y]^v =$$
$$= t(X^c, Y^c) - \frac{1}{2}\big[[X^v, \operatorname{grad}\beta^v], Y^v\big] + \frac{1}{2}\big[[Y^v, \operatorname{grad}\beta^v], X^v\big] \overset{\text{Th. 4.2 (c)}}{=}$$
$$= -\frac{1}{2}\big[[X^v, \operatorname{grad}\beta^v], Y^v\big] + \frac{1}{2}\big[[Y^v, \operatorname{grad}\beta^v], X^v\big] = 0$$

using the Jacobi identity. This means that $\widetilde{h}$ is torsion-free.

On the other hand, for any vector field $X \in \mathfrak{X}(TM)$,

$$d_{\widetilde{h}}E(X) = \widetilde{h}(X)E = h(X)E - \frac{1}{2}[J, \operatorname{grad}\beta^v](X)E \overset{\text{Th. 4.2 (a)}}{=}$$
$$= -\frac{1}{2}[J, \operatorname{grad}\beta^v](X)E \overset{\text{II.(1.2d)}}{=} \mathcal{C}(F\operatorname{grad}\beta^v, X)E = dE\big(\mathcal{C}(F\operatorname{grad}\beta^v, X)\big) =$$
$$\overset{(4.1)}{=} -\omega\big(S, \mathcal{C}(F\operatorname{grad}\beta^v, X)\big) = \omega(\mathcal{C}(F\operatorname{grad}\beta^v, X), S) =$$
$$= g(\mathcal{C}(F\operatorname{grad}\beta^v, X), C) = \mathcal{C}_\flat(F\operatorname{grad}\beta^v, X, S) \overset{\text{Lemma 3.8}}{=} 0,$$

i.e. $\widetilde{h}$ is conservative.

4.7. Corollary. *Let (M, E) be a Finsler manifold and h the Barthel endomorphism. If $\mathcal{C}'$ is the second Cartan tensor belonging to h, then $i_S\mathcal{C}' = 0$, for any semispray S.*

Proof. We consider the classical Cartan connection (D, h). Then for each vector field $X \in \mathfrak{X}(\mathcal{T}M)$:

$$\begin{aligned} 0 &\overset{(4.5h)}{=} D_{hX}C = D_{hX}JS \overset{(4.5e)}{=} v[hX, C] + \mathcal{C}'(X, S) = \\ &= H(hX) + \mathcal{C}'(X, S) = 2(h^*DC)(hX) + \mathcal{C}'(X, S) \overset{(4.5h)}{=} \mathcal{C}'(X, S). \qquad \square \end{aligned}$$

4.8. Lemma. *Let (M, E) be a Finsler manifold, (D, h) the classical Cartan connection and S the canonical spray. Then*

(4.8a) $$D_S\mathcal{C} = -\mathcal{C}';$$

(4.8b) $$D_C\mathcal{C} = -\mathcal{C}, \qquad D_C\mathcal{C}' = 0,$$

where $\mathcal{C}'$ is the second Cartan tensor belonging to h.

For a proof see [11], pp. 331–332 and 335.

4.9. Theorem and definition. *Let (M, E) be a Finsler manifold and h a conservative torsion-free horizontal endomorphism on M. Assume g_h is the prolongation of g along h and $\mathcal{C}'$ is the second Cartan tensor belonging to h. There exists a unique Finsler connection $(\overset{\mathrm{R}}{D}, h)$ on M such that*

(4.9a) $$J^*\overset{\mathrm{R}}{D} = J^*\overset{\circ}{D};$$

(4.9b) *$\overset{\mathrm{R}}{D}$ is h-metrical, i.e. $\forall X \in \mathfrak{X}(\mathcal{T}M) : \overset{\mathrm{R}}{D}_{hX}g_h = 0$;*

(4.9c) *the $(h)h$-torsion of $\overset{\mathrm{R}}{D}$ vanishes.*

The covariant derivatives with respect to $\overset{\mathrm{R}}{D}$ can be calculated by the following formulas: for each vector fields X, Y on $\mathcal{T}M$,

(4.9d) $$\overset{\mathrm{R}}{D}_{JX}JY = J[JX, Y] = \overset{\circ}{D}_{JX}JY;$$

(4.9e) $$\overset{\mathrm{R}}{D}_{hX}JY = v[hX, JY] + \mathcal{C}'(X, Y) = D_{hX}JY;$$

(4.9f) $$\overset{\mathrm{R}}{D}_{JX}hY = h[JX, Y] = \overset{\circ}{D}_{JX}hY;$$

(4.9g) $$\overset{\mathrm{R}}{D}_{hX}hY = hF[hX, JY] + F\mathcal{C}'(X, Y) = D_{hX}hY.$$

Then

$$h^*\overset{\mathrm{R}}{D}C = \frac{1}{2}H.$$

Therefore, if in addition to (4.9a)–(4.9c)

$$h^*\overset{\mathrm{R}}{D}C = 0 \tag{4.9h}$$

is also satisfied, then h is the Barthel endomorphism of the Finsler manifold. In this case $(\overset{\mathrm{R}}{D}, h)$ is called the (classical) Rund *(or the* Chern – Rund) connection*.*

The proof of this theorem is completely analogous to that of Theorem 4.5.

4.10. Remark. The classical Chern – Rund connection was first constructed by S. S. Chern in 1948, using a local coframe field. Three years later H. Rund also discovered an important, seemingly different Finsler connection. Almost fifty years had passed until M. Anastasiei [1] realized that both constructions result the *same* Finsler connection.

4.11. Remark. Using vertically and horizontally lifted vector fields, the rules of calculation for covariant derivatives take a somewhat simpler form. Namely, if $\widetilde{D}$ stands for $\overset{\circ}{D}$, D or $\overset{\mathrm{R}}{D}$, we get the following table:

	Berwald $(\overset{\circ}{D})$	Cartan (D)	Rund $(\overset{\mathrm{R}}{D})$
$\widetilde{D}_{X^v}Y^v$	0	$\mathcal{C}(X^h, Y^h)$	0
$\widetilde{D}_{X^h}Y^v$	$[X^h, Y^v]$	$[X^h, Y^v] + \mathcal{C}'(X^h, Y^h)$	$[X^h, Y^v] + \mathcal{C}'(X^h, Y^h)$
$\widetilde{D}_{X^v}Y^h$	0	$F\mathcal{C}(X^h, Y^h)$	0
$\widetilde{D}_{X^h}Y^h$	$F[X^h, Y^v]$	$F[X^h, Y^v] + F\mathcal{C}'(X^h, Y^h)$	$F[X^h, Y^v] + F\mathcal{C}'(X^h, Y^h)$

5. Basic curvature identities

5.1. Convention. Throughout this section (M, E) is a Finsler manifold and h is the Barthel endomorphism on M. $\overset{\circ}{D}$, D and $\overset{\mathrm{R}}{D}$ denote the Berwald, the (classical) Cartan and the Rund connection, respectively.

5.2. Proposition. *Let $\overset{\circ}{\mathbb{R}}$, $\mathbb{R}$ and $\overset{\mathrm{R}}{\mathbb{R}}$ be the h-curvature of the Berwald, the Cartan and the Rund connection, respectively. We have the following relations:*

$$
\begin{aligned}
\text{(5.2a)}\qquad \mathbb{R}(X,Y)Z = {} & \overset{\circ}{\mathbb{R}}(X,Y)Z + (D_{hX}\mathcal{C}')\,(Y,Z) - (D_{hY}\mathcal{C}')\,(X,Z) - \\
& - \mathcal{C}'(X, F\mathcal{C}'(Y,Z)) + \mathcal{C}'(Y, F\mathcal{C}'(X,Z)) + \\
& + \mathcal{C}(F\Omega(X,Y), Z);
\end{aligned}
$$

$$
\text{(5.2b)}\qquad \overset{\mathrm{R}}{\mathbb{R}}(X,Y)Z = \mathbb{R}(X,Y)Z - \mathcal{C}(F\Omega(X,Y),Z) \qquad (X,Y,Z \in \mathfrak{X}(\mathcal{T}M)).
$$

Proof. The first formula has been obtained by J. Grifone; see [11], pp.333–334. Since the tensors $\mathbb{R}$ and $\overset{\mathrm{R}}{\mathbb{R}}$ are semibasic, it is enough to check (5.2b) for triplets of the form (X^c, Y^c, Z^c); $X, Y, Z \in \mathfrak{X}(M)$. Taking into account that $\overset{\mathrm{R}}{D}{}^h = D^h$ from (4.9e) and (4.9g) while $\overset{\mathrm{R}}{D}{}^v = \overset{\circ}{D}{}^v$ from (4.9d) and (4.9f), we get:

$$
\begin{aligned}
\overset{\mathrm{R}}{\mathbb{R}}(X^c,Y^c)Z^c &= \overset{\mathrm{R}}{\mathbb{K}}(hX^c,hY^c)JZ^c = \overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{Y^h}Z^v - \overset{\mathrm{R}}{D}_{Y^h}\overset{\mathrm{R}}{D}_{X^h}Z^v - \overset{\mathrm{R}}{D}_{[X^h,Y^h]}Z^v = \\
&= D_{X^h}D_{Y^h}Z^v - D_{Y^h}D_{X^h}Z^v - \overset{\mathrm{R}}{D}_{h[X^h,Y^h]}Z^v - \overset{\mathrm{R}}{D}_{v[X^h,Y^h]}Z^v = \\
&= \mathbb{R}(X^c,Y^c)Z^c + D_{v[X^h,Y^h]}Z^v - \overset{\mathrm{R}}{D}_{v[X^h,Y^h]}Z^v = \mathbb{R}(X^c,Y^c)Z^c \\
&\quad + \mathcal{C}(F[X^h,Y^h],Z^c) = \mathbb{R}(X^c,Y^c)Z^c + \mathcal{C}(Fv[X^h,Y^h],Z^c) + \\
&\quad + \mathcal{C}(Fh[X^h,Y^h],Z^c) \overset{(2.7\mathrm{a}),(2.6\mathrm{d})}{=} \mathbb{R}(X^c,Y^c)Z^c - \\
&\quad - \mathcal{C}(F\Omega(X^c,Y^c),Z^c).
\end{aligned}
$$

□

5.3. Proposition. *The hv-curvature tensors of the Cartan, the Berwald and the Rund connection are related as follows:*

$$
\begin{aligned}
\text{(5.3a)}\qquad \mathbb{P}(X,Y)Z = {} & \overset{\circ}{\mathbb{P}}(X,Y)Z + (D_{hX}\mathcal{C})\,(hY,hZ) - (D_{JY}\mathcal{C}')\,(hX,hZ) + \\
& + \mathcal{C}(F\mathcal{C}'(hX,hY),hZ) + \mathcal{C}(F\mathcal{C}'(hX,hZ),hY) - \\
& - \mathcal{C}'(F\mathcal{C}(hX,hY),hZ) - \mathcal{C}'(F\mathcal{C}(hY,hZ),hX).
\end{aligned}
$$

$$
\text{(5.3b)}\qquad \overset{\mathrm{R}}{\mathbb{P}}(X^c,Y^c)Z^c = \overset{\circ}{\mathbb{P}}(X^c,Y^c)Z^c + [\mathcal{C}'(X^h,Z^h),Y^v].
$$

Proof. In the same manner as above, evaluating $\mathbb{P}$ on a triplet (X^c, Y^c, Z^c) we get its expression in terms of the hv-curvature tensor of the Berwald connection, the

Cartan tensors and their h- or v-covariant derivatives, respectively (for the details see e.g. [36], p. 52). A similar reasoning, but a shorter calculation shows that

$$\begin{aligned}\overset{\mathrm{R}}{\mathbb{P}}(X^c, Y^c)Z^c &= \overset{\mathrm{R}}{\mathbb{K}}(X^h, Y^v)Z^v = \overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{Y^v}Z^v - \overset{\mathrm{R}}{D}_{Y^v}\overset{\mathrm{R}}{D}_{X^h}Z^v - \overset{\mathrm{R}}{D}_{[X^h,Y^v]}Z^v = \\ &= -\overset{\mathrm{R}}{D}_{Y^v}\big([X^h, Z^v] + \mathcal{C}'(X^h, Z^h)\big) - \overset{\mathrm{R}}{D}_{JF[X^h,Y^v]}JZ^h = \\ &= -\overset{\mathrm{R}}{D}_{JY^h}JF[X^h, Z^v] - \overset{\mathrm{R}}{D}_{JY^h}JF\mathcal{C}'(X^h, Z^h) - J\big[[X^h, Y^v], Z^h\big].\end{aligned}$$

Taking into account that for any vector fields $X, Y, Z \in \mathfrak{X}(M)$

$$[X^h, Y^v],\ \big[[X^h, Y^v], Z^h\big], \dots \text{ are vertical,}$$

we obtain that

$$\begin{aligned}\overset{\mathrm{R}}{\mathbb{P}}(X^c, Y^c)Z^c &= -\overset{\mathrm{R}}{D}_{JY^h}JF[X^h, Z^v] - \overset{\mathrm{R}}{D}_{JY^h}JF\mathcal{C}'(X^h, Z^h) = \\ &= -J\big[Y^v, F[X^h, Z^v]\big] - J[Y^v, F\mathcal{C}'(X^h, Z^h)] = \\ &= \overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c + [\mathcal{C}'(X^h, Z^h), Y^v],\end{aligned}$$

since

$$0 \overset{(1.16c)}{=} [J, Y^v]F\mathcal{C}'(X^h, Z^h) \overset{(2.8a)}{=} [\mathcal{C}'(X^h, Z^h), Y^v] - J[F\mathcal{C}'(X^h, Z^h), Y^v]. \qquad \square$$

5.4. Proposition.
(a) *The v-curvature tensor of the Berwald and of the Rund connection vanishes, i.e., $\overset{\circ}{\mathbb{Q}} = 0$ and $\overset{\mathrm{R}}{\mathbb{Q}} = 0$.*
(b) *For the v-curvature tensor of the Cartan connection we have the expression*

$$\text{(5.4)} \qquad \mathbb{Q}(X, Y)Z = \mathcal{C}\big(F\mathcal{C}(Z, X), Y\big) - \mathcal{C}\big(X, F\mathcal{C}(Y, Z)\big) \qquad (X, Y, Z \in \mathfrak{X}(\mathcal{T}M)).$$

Proof. The first assertion can be verified by an immediate calculation. Formula (5.4) was derived by J. Grifone [11], we recall only the key observation, the identity

$$\text{(5.4a)} \qquad (D_{JX}\mathcal{C})(Y, Z) = (D_{JY}\mathcal{C})(X, Z) \qquad (X, Y, Z \in \mathfrak{X}(\mathcal{T}M)). \qquad \square$$

5.5. Example. Suppose that (M, E) is a positive definite two-dimensional Finsler manifold. Let S be the canonical spray, h the Barthel endomorphism and F the almost complex structure induced by h. Consider the prolongation g_h of the Riemann – Finsler metric along h (3.2b). Let $C_0 := \frac{1}{\sqrt{2E}}C$ be the normalized Liouville vector field; then $g_h(C_0, C_0) = 1$. The vector field $S_0 := \frac{1}{\sqrt{2E}}S = FC_0$ is g_h-orthogonal to C_0, i.e.,

$$g_h(S_0, C_0) = g_h(hS_0, vC_0) = g_h(hS_0, JFC_0) \overset{(3.2c)}{=} 0,$$

and $g_h(S_0, S_0) = 1$. Next, using the Gram – Schmidt process, we can construct — at least locally — a g_h-orthonormal basis (C_0, X_0) of $\mathfrak{X}^v(\mathcal{T}M)$, where the vector field X_0 is uniquely determined up to sign. Using the almost complex structure once more, we arrive at a (local) g_h-orthonormal basis

$$(C_0, X_0, FX_0, S_0) \tag{5.5a}$$

for the module $\mathfrak{X}(\mathcal{T}M)$. The quadruple (5.5a) is called the *Berwald frame* of the Finsler manifold after L. Berwald, see his posthumous paper [3]. As for the details of the coordinate-free construction, we refer to [39].

Now, since the first Cartan tensor $\mathcal{C}$ is semibasic, it follows by 3.8 that $\mathcal{C}$ is completely determined by its value on the pair (FX_0, FX_0). Taking into account (5.4), we infer immediately that *in two dimensions the v-curvature of the Cartan connection vanishes.* This proposition was first proved by D. Laugwitz [20] with the machinery of classical tensor calculus.

5.6. Remark. If $n \geq 3$, (M, E) is a positive-definite n-dimensional Finsler manifold, and the energy function is symmetric ($E(-v) = E(v)$ for any tangent vector $v \in \mathcal{T}M$), then the vanishing of $\mathbb{Q}$ implies that (M, E) is a Riemannian manifold. This far from trivial result (conjectured by D. Laugwitz [21]) was first proved by F. Brickell [4].

6. Berwald manifolds

6.1. Convention. *Retaining the notations introduced above, in our following discussion h will denote the Barthel endomorphism and $\mathcal{C}'$ the second Cartan tensor belonging to h, unless otherwise stated.*

6.2. To begin with, we recall an important classical result, first formulated and proved *intrinsically* by J. G. Diaz; see [8].

6.3. Theorem and definition. *Let (M, E) be a Finsler manifold. The following assertions are equivalent:*

(a) *The hv-curvature $\mathbb{P}$ of the Cartan connection vanishes.*

(b) *The second Cartan tensor $\mathcal{C}'$ vanishes.*

(c) $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : (D_{hX}\mathcal{C})(Y, Z) = (D_{hZ}\mathcal{C})(X, Y)$.

(d) $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : \overset{\circ}{\mathbb{P}}(X, Y)Z = -(D_{hX}\mathcal{C})(Y, Z)$.

If one, and therefore all, of the conditions (a)–(d) are satisfied, then (M, E) is called a Landsberg manifold.

6.4. Other characterizations.

(a) The property $\mathcal{C}' = 0$ implies immediately (see e.g. 4.11) that in a Landsberg manifold the h-covariant derivatives with respect to the Cartan, the Berwald and the Rund connection coincide. Conversely, if $D^h = \overset{\circ}{D}{}^h$ or $\overset{\circ}{D}{}^h = \overset{\mathrm{R}}{D}{}^h$, then $\mathcal{C}' = 0$ and the Finsler manifold is a Landsberg manifold.

(b) Now let us have a look at the $(v)hv$-torsion of the Cartan connection. For each vector fields X, Y on M,

$$\begin{aligned}\mathbb{P}^1(X^h, Y^v) &:= v\mathbb{T}(X^h, Y^v) = vD_{X^h}Y^v - vD_{Y^v}X^h - v[X^h, Y^v] = \\ &\overset{4.11}{=} v[X^h, Y^v] + v\mathcal{C}'(X^h, Y^h) - vF\mathcal{C}(X^h, Y^h) - \\ &\quad - v[X^h, Y^v] \overset{(2.8b)}{=} \mathcal{C}'(X^h, Y^h) + J\mathcal{C}(X^h, Y^h) = \mathcal{C}'(X^h, Y^h).\end{aligned}$$

It follows that the *vanishing of the $(v)hv$-torsion of the Cartan connection characterizes the Landsberg manifolds.*

(c) We infer immediately from (3.6b) that *a Finsler manifold is a Landsberg manifold if and only if the Berwald connection is h-metrical* (i.e., $\overset{\circ}{D}{}^h g_h = 0$). In Matsumoto's monograph [24] Landsberg manifolds are defined by this property.

6.5. Definition. A Finsler manifold (M, E) is said to be a *Berwald manifold* if there is a linear connection ∇ on M such that for each vector fields X, Y on M,

$$(\nabla_X Y)^v = [X^h, Y^v],$$

where the horizontal lifting is taken with respect to the Barthel endomorphism.

6.6. Remarks.

(a) The linear connection ∇ in definition 6.5 is clearly unique, so it will be mentioned as *the linear connection of the Berwald manifold.* One can see also at once that the horizontal endomorphism induced by ∇ is just the Barthel endomorphism.

We immediately infer that *the Barthel endomorphism and the canonical spray of a Berwald manifold are smooth on the whole tangent manifold* TM. The converse is also true: if the canonical spray of the Finsler manifold (M,E) is smooth on TM, then (M,E) is a Berwald manifold. For another reasoning see [8].

(b) By a clever observation of Z. I. Szabó the linear connection ∇ of a positive definite Berwald manifold is *Riemann – metrizable*: there always exists a Riemannian metric g_M on M whose Levi–Civita connection is ∇. This is the first step toward the classification of positive definite Berwald manifolds achieved by him in [32].

(c) As in Example 3.10, consider a Randers manifold (M,E) with energy function $E = \frac{1}{2}L^2$; $L := L_\alpha + \widetilde{\beta}$. It is well-known (see e.g. [18], [23] and [27]) that *(M,E) is a Berwald manifold if and only if β is parallel with respect to the Lévi – Civita connection of the metric α.*

In this case the Lévi – Civita connection coincides with the linear connection of (M,E) as a Berwald manifold.

6.7. Lemma. *A Finsler manifold (M,E) is a Berwald manifold if and only if*

$$\forall X,Y \in \mathfrak{X}(M) : [X^h, Y^v] \quad \textit{is a vertical lift.} \tag{6.7}$$

Proof. The necessity of (6.7) is evident. To see the sufficiency, we consider the mapping

$$\nabla : (X,Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \nabla_X Y \in \mathfrak{X}(M), (\nabla_X Y)^v := [X^h, Y^v],$$

where the horizontal lifting is taken with respect to the Barthel endomorphism, of course. Then ∇ is well-defined and an easy calculation shows that it is a linear connection, indeed. $\square$

6.8. Lemma. *Suppose that (M,E) is a Berwald manifold and let ∇ be its linear connection. Then the pair $(\overset{h}{\nabla}, h)$, where $\overset{h}{\nabla}$ is the horizontal lift of ∇ and h is the Barthel endomorphism, is just the Berwald connection.*

Proof. Our only task is to check (4.3a) and (4.3b). But this is easy: for any vector fields X, Y, Z on M, we have

$$\mathbb{P}^1(X^h, Y^v) = v\left(\overset{h}{\nabla}_{X^h} Y^v - \overset{h}{\nabla}_{Y^v} X^h - [X^h, Y^v]\right) \overset{(2.13)}{=} v\left((\nabla_X Y)^v - [X^h, Y^v]\right) = 0,$$

$$\mathbb{B}(X^h, Y^v) = h\left(\overset{h}{\nabla}_{X^h} Y^v - \overset{h}{\nabla}_{Y^v} X^h - [X^h, Y^v]\right) \overset{(2.13)}{=} 0. \qquad \square$$

6.9. A coordinate view. Let $\left(\mathcal{U}, (u^i)_{i=1}^n\right)$ be a chart on M. With the help of the induced chart

$$\left(\pi^{-1}(\mathcal{U}), (x^i, y^i)_{i=1}^n\right); \quad x^i := u^i \circ \pi,$$
$$y^i : v \in \pi^{-1}(\mathcal{U}) \mapsto y^i(v) := v(u^i) \qquad (1 \le i \le n)$$

we review some important coordinate expressions. Einstein's summation convention will be used.

(i) We get from (3.3b) that the components of the Riemann – Finsler metric g are

$$g_{ij} := g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial^2 E}{\partial y^i \partial y^j}.$$

(3.6a) implies that the components of the first Cartan tensors $\mathcal{C}_\flat$ and $\mathcal{C}$ are

$$(\mathcal{C}_\flat)_{ijk} = \frac{1}{2}C_{ijk}, \qquad \mathcal{C}^\ell_{ij} = \frac{1}{2}C^\ell_{ij}$$

where

$$C_{ijk} := \frac{\partial^3 E}{\partial y^i \partial y^j \partial y^k}, \quad C^\ell_{ij} := g^{\ell k} C_{ijk}, \quad \left(g^{\ell k}\right) := \left(g_{\ell k}\right)^{-1}.$$

The coordinate expression of the canonical spray S is

$$S \restriction \pi^{-1}(\mathcal{U}) = y^k \frac{\partial}{\partial x^k} - 2G^k \frac{\partial}{\partial y^k},$$

where

$$G^k = g^{jk} G_j, \quad G_j := \frac{1}{2}\left(y^k \frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j}\right). \tag{6.9a}$$

The Barthel endomorphism can be represented in the form

$$h \restriction \mathfrak{X}\left(\pi^{-1}(\mathcal{U})\right) = \left(\frac{\partial}{\partial x^i} - G^k_i \frac{\partial}{\partial y^k}\right) \otimes dx^i, \quad G^k_i := \frac{\partial G^k}{\partial y^i} \qquad (1 \le i, k \le n). \tag{6.9b}$$

The Berwald connection $(\overset{\circ}{D}, h)$ of (M, E) is completely determined by the functions

$$G^k_{ij} := \frac{\partial G^k_i}{\partial y^j} = \frac{\partial^2 G^k}{\partial y^i \partial y^j} \qquad (1 \le i, j, k \le n). \tag{6.9c}$$

The functions G^k_{ij} are called the *connection parameters* for the Berwald connection.

(ii) Traditionally Berwald manifolds are defined as follows: *"the connection parameters for the Berwald connection depend only on the position"*, i.e. by the condition

$$G^k_{ij\ell} := \frac{\partial G^k_{ij}}{\partial y^\ell} = 0 \qquad (1 \le i, j, k, \ell \le n).$$

Finsler manifolds with this property were called *affinely connected spaces* by L. Berwald himself, see [2]. Now we show that our definition is equivalent to the classical one. To see this, let ∇ be a linear connection on M, locally given by the functions $\Gamma^k_{ij} \in C^\infty(\mathcal{U})$ $(1 \le i, j, k \le n)$, such that

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \Gamma^k_{ij} \frac{\partial}{\partial u^k}.$$

Then

$$\left(\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j}\right)^v = \left(\Gamma^k_{ij}\circ\pi\right)\frac{\partial}{\partial y^k},$$

$$\left[\left(\frac{\partial}{\partial u^i}\right)^h,\left(\frac{\partial}{\partial u^j}\right)^v\right] = \left[h\left(\frac{\partial}{\partial u^i}\right)^c,\frac{\partial}{\partial y^j}\right] \overset{(6.9\mathrm{b})}{=} \left[\frac{\partial}{\partial x^i} - G^k_i\frac{\partial}{\partial y^k},\frac{\partial}{\partial y^j}\right] =$$
$$= \frac{\partial G^k_i}{\partial y^j}\frac{\partial}{\partial y^k} \overset{(6.9\mathrm{c})}{=} G^k_{ij}\frac{\partial}{\partial y^k},$$

thus

$$\forall i,j\in\{1,\dots,n\} : \left(\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j}\right)^v = \left[\left(\frac{\partial}{\partial u^i}\right)^h,\left(\frac{\partial}{\partial u^j}\right)^v\right]$$

$$\iff \Gamma^k_{ij}\circ\pi = G^k_{ij} \qquad (1\le k\le n)$$
$$\iff G^k_{ij\ell} = 0 \qquad (1\le k,\ell\le n).$$

(iii) Notice that the components of the hv-curvature tensor $\overset{\circ}{\mathbb{P}}$ of the Berwald connection are just the functions $-G^k_{ij\ell}$ (hence the classical definition in (ii) has a tensorial character). Indeed,

$$\overset{\circ}{\mathbb{P}}\left(\left(\frac{\partial}{\partial u^i}\right)^h,\left(\frac{\partial}{\partial u^j}\right)^h\right)\left(\frac{\partial}{\partial u^\ell}\right)^h \overset{(2.15\mathrm{b})}{=} \left[\left[\left(\frac{\partial}{\partial u^i}\right)^h,\left(\frac{\partial}{\partial u^j}\right)^v\right],\left(\frac{\partial}{\partial u^\ell}\right)^v\right] =$$
$$= \left[G^k_{ij}\frac{\partial}{\partial y^k},\frac{\partial}{\partial y^\ell}\right] = -\frac{\partial G^k_{ij}}{\partial y^\ell} = -G^k_{ij\ell}.$$

It follows that *a Finsler manifold is a Berwald manifold if and only if the hv-curvature of the Berwald connection vanishes*. We shall see soon how this follows intrinsically at once.

6.10. Proposition. *The second Cartan tensor $\mathcal{C}'$ vanishes in any Berwald manifold, consequently the Berwald manifolds are at the same time Landsberg manifolds.*

Proof. We have already seen in the proof of 3.9 that for any vector fields X, Y, Z on M,

$$2\mathcal{C}'_\flat(X^c,Y^c,Z^c) = \left[Y^v,[X^h,Z^v]\right]E$$

(due to the fact that h is homogeneous and conservative). Since (M,E) is a Berwald manifold, $[X^h,Z^v]$ is a vertical lift and, consequently $[Y^v,[X^h,Z^v]] = 0$ by Lemma 1.16, whence our conclusion. □

6.11. Corollary. *The Berwald connection of a Berwald manifold is h-metrical (i.e., $\overset{\circ}{D}{}^h g_h = 0$).*

6.12. Theorem. *Let (M,E) be a Finsler manifold. Then the following assertions are equivalent:*

(a) *(M,E) is a Berwald manifold.*

(b) *The hv-curvature tensor $\overset{\circ}{\mathbb{P}}$ of the Berwald connection vanishes.*

(c) *The hv-curvature tensor $\overset{\mathrm{R}}{\mathbb{P}}$ of the Rund connection vanishes.*

(d) *With respect to the Cartan connection, the h-covariant derivative of the first Cartan tensor vanishes (i.e., $D^h\mathcal{C}=0$).*

Proof.

(a) $\Longleftrightarrow$ (b) This is an immediate consequence of (2.15b), Lemma 6.7 and Lemma 1.14.

(a) $\Longrightarrow$ (c) We infer this at once from (5.3b), from Proposition 6.10 ($\mathcal{C}'=0$) and from the equivalence (a) $\Longleftrightarrow$ (b).

(c) $\Longrightarrow$ (a) To prove the implication, note first that

$$\forall X,Y \in \mathfrak{X}(M) : \overset{\mathrm{R}}{\mathbb{P}}(X^h,Y^h)S = \mathcal{C}'(X^h,Y^h),$$

where S is an arbitrary semispray. Indeed,

$$\overset{\mathrm{R}}{\mathbb{P}}(X^h,Y^h)S = \overset{\mathrm{R}}{\mathbb{K}}(X^h,Y^v)C = \overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{Y^v}C - \overset{\mathrm{R}}{D}_{Y^v}\overset{\mathrm{R}}{D}_{X^h}C - \overset{\mathrm{R}}{D}_{[X^h,Y^v]}C,$$

where $\overset{\mathrm{R}}{D}_{X^h}C = 0$ by (4.9h), while

$$\begin{aligned}\overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{Y^v}C &= \overset{\mathrm{R}}{D}_{X^h}\overset{\mathrm{R}}{D}_{JY^h}JS \overset{(4.9d)}{=} \overset{\mathrm{R}}{D}_{X^h}J[Y^v,S] \overset{(1.15)}{=} \overset{\mathrm{R}}{D}_{X^h}Y^v = \\ &\overset{4.11}{=} [X^h,Y^v] + \mathcal{C}'(X^h,Y^h),\\ \overset{\mathrm{R}}{D}_{[X^h,Y^v]}C &\overset{(4.9d),(1.15)}{=} [X^h,Y^v],\end{aligned}$$

thus we obtain the desired relation. We conclude that $\overset{\mathrm{R}}{\mathbb{P}}=0$ implies that $\mathcal{C}'=0$ and, therefore, that $\overset{\circ}{\mathbb{P}}=0$ (by (5.3b)); hence our assertion.

(a) $\Longrightarrow$ (d) In view of the property $\mathcal{C}'=0$ it follows that in Berwald manifolds $D^h = \overset{\circ}{D}{}^h$ (cf. 6.4(a)). Therefore it is enough to check that

$$\forall X \in \mathfrak{X}(M) : \overset{\circ}{D}_{X^h}\mathcal{C} = 0.$$

Let $X,Y,Z,U \in \mathfrak{X}(M)$ be arbitrary. $\overset{\circ}{D}$ is h-metrical by Corollary 6.11, so

$$\begin{aligned}0 = \big(\overset{\circ}{D}_{X^h}g_h\big)\big(\mathcal{C}(Y^h,Z^h),U^v\big) &= X^h g_h\big(\mathcal{C}(Y^h,Z^h),U^v\big) - g_h\big(\overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h),U^v\big) - \\ - g_h\big(\mathcal{C}(Y^h,Z^h),\overset{\circ}{D}_{X^h}U^v\big) &\overset{(3.4a),4.11}{=} \frac{1}{2}X^h(\mathcal{L}_{JY^h}J^*g_h)(Z^h,U^h) - \\ - g_h\big(\overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h),U^v\big) &- \frac{1}{2}\big(\mathcal{L}_{JY^h}J^*g_h\big)(Z^h,F[X^h,U^v])\end{aligned}$$

and hence

$$\begin{aligned}2g_h\big(\overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h),U^v\big) &= X^h\big(Y^v g_h(Z^v,U^v) - g_h(J[Y^v,Z^h],U^v) - \\ &- g_h(Z^v,J[Y^v,U^h])\big) - Y^v g_h(Z^v,[X^h,U^v]) + \\ &+ g_h(J[Y^v,Z^h],[X^h,U^v]) + g_h\big(Z^v,J[Y^v,F[X^h,U^v]]\big) = \\ &\overset{(1.8a)}{=} X^h\big(Y^v g_h(Z^v,U^v)\big) - Y^v g_h(Z^v,[X^h,U^v]),\end{aligned}$$

taking into account that from the proof of (2.15b)

$$J\big[Y^v,F[X^h,U^v]\big] = -J\big[F[X^h,U^v],Y^v\big] = -\overset{\circ}{\mathbb{P}}(X^c,Y^c)U^c \overset{(a)\iff(b)}{=} 0.$$

Now we evaluate the term $\overset{\circ}{D}_{X^h}\mathcal{C}$. For each vector fields Y,Z on M,

$$\begin{aligned}\big(\overset{\circ}{D}_{X^h}\mathcal{C}\big)(Y^h,Z^h) &= \overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h) - \mathcal{C}(\overset{\circ}{D}_{X^h}Y^h,Z^h) - \mathcal{C}(Y^h,\overset{\circ}{D}_{X^h}Z^h) = \\ &\overset{4.11}{=} \overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h) - \mathcal{C}(F[X^h,Y^v],Z^h) - \mathcal{C}(Y^h,F[X^h,Z^v]).\end{aligned}$$

Applying the last two results, we obtain:

$$\begin{aligned}2g_h\Big(\big(\overset{\circ}{D}_{X^h}\mathcal{C}\big)(Y^h,Z^h),U^v\Big) &= 2g_h\big(\overset{\circ}{D}_{X^h}\mathcal{C}(Y^h,Z^h),U^v\big) - \\ &- 2g_h\big(\mathcal{C}(F[X^h,Y^v],Z^h),U^v\big) - 2g_h\big(\mathcal{C}(Y^h,F[X^h,Z^v]),U^v\big) = \\ &= X^h\big(Y^v g_h(Z^v,U^v)\big) - Y^v g_h(Z^v,[X^h,U^v]) - \big(\mathcal{L}_{[X^h,Y^v]}J^*g_h\big)(Z^h,U^h) - \\ &\big(\mathcal{L}_{Y^v}J^*g_h\big)(F[X^h,Z^v],U^h) = X^h\big(Y^v g_h(Z^v,U^v)\big) - Y^v g_h(Z^v,[X^h,U^v]) - \\ &- [X^h,Y^v]g_h(Z^v,U^v) + g_h\big(J\big[[X^h,Y^v],Z^h\big],U^v\big) + g_h\big(Z^v,J\big[[X^h,Y^v],U^h\big]\big) \\ &- Y^v g_h([X^h,Z^v],U^v) + g_h\big(J\big[Y^v,F[X^h,Z^v]\big],U^v\big) + \\ &+ g_h([X^h,Z^v],J[Y^v,U^h]) = -Y^v g_h(Z^v,[X^h,U^v]) + Y^v\big(X^h g_h(Z^v,U^v)\big) - \\ &- Y^v g_h([X^h,Z^v],U^v) - g_h\big(\overset{\circ}{\mathbb{P}}(X^c,Y^c)Z^c,U^v\big) = Y^v\big(X^h g_h(Z^v,U^v)\big) - \\ &- Y^v g_h([X^h,Z^v],U^v) - Y^v g_h(Z^v,[X^h,U^v]) \overset{4.11}{=} Y^v\Big[\Big(\overset{\circ}{D}_{X^h}g_h\Big)(Z^v,U^v)\Big] = 0,\end{aligned}$$

since $\overset{\circ}{D}$ is h-metrical. Hence $\overset{\circ}{D}_{hX}\mathcal{C} = 0$.

(d) $\Longrightarrow$ (a) Assume $D^h\mathcal{C} = 0$. Then for each vector field X on $\mathcal{T}M$, $D_{hX}\mathcal{C} = 0$. In particular, taking the canonical spray S, we obtain:

$$0 = D_{hS}\mathcal{C} = D_S\mathcal{C} \overset{(4.8.a)}{=} -\mathcal{C}'.$$

The vanishing of $\mathcal{C}'$ implies by Theorem 6.3 that

$$\forall X,Y,Z \in \mathfrak{X}(\mathcal{T}M) : \overset{\circ}{\mathbb{P}}(X,Y)Z = -\left(D_{hX}\mathcal{C}\right)(Y,Z) \overset{(d)}{=} 0,$$

so (M,E) is a Berwald manifold. With this we reach the end of the proof of 6.12. □

7. Locally Minkowski manifolds

7.1. Proposition and definition. *Let (M, E) be a Berwald manifold. Then the following conditions are equivalent:*

$$\text{(a)}\quad \Omega := -\frac{1}{2}[h,h] = 0, \qquad \text{(b)}\quad \mathbb{R} = 0,$$

$$\text{(c)}\quad \overset{\circ}{\mathbb{R}} = 0, \qquad \text{(d)}\quad \overset{\mathrm{R}}{\mathbb{R}} = 0.$$

If one, and therefore all, of these conditions are satisfied, then (M, E) is called a *locally Minkowski manifold.*

Proof.

(a) $\Longleftrightarrow$ (b) Since the second Cartan tensor $\mathcal{C}'$ vanishes in any Berwald manifold, formula (5.2a) reduces to

$$\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : \mathbb{R}(X,Y)Z = \overset{\circ}{\mathbb{R}}(X,Y)Z + \mathcal{C}(F\Omega(X,Y), Z) \tag{7.1}$$

in this case.We infer at once from (7.1) and 2.18 that (a) $\Longrightarrow$ (b). Conversely, suppose that $\mathbb{R} = 0$ and let S be the canonical spray. Then for each vector fields X, Y on $\mathcal{T}M$,

$$0 \overset{(7.1)}{=} \overset{\circ}{\mathbb{R}}(X,Y)S + \mathcal{C}(F\Omega(X,Y), S) \overset{3.8}{=} \overset{\circ}{\mathbb{R}}(X,Y)S \overset{(2.17)}{=} \Omega(X,Y),$$

so (b) $\Longrightarrow$ (a).

(b) $\Longleftrightarrow$ (c) We have just seen that $\mathbb{R} = 0$ implies $\Omega = 0$. Then, by (7.1), $\overset{\circ}{\mathbb{R}} = 0$; thus we get the desired implication (b) $\Longrightarrow$ (c). The converse is obvious from (7.1) and (2.17).

(b) $\Longleftrightarrow$ (d) If $\mathbb{R} = 0$, then $\Omega = 0$ by the equivalence (a) $\Longleftrightarrow$ (b) and we deduce from (5.2b) that $\overset{\mathrm{R}}{\mathbb{R}} = 0$. Thus (b) $\Longrightarrow$ (d). Assume $\overset{\mathrm{R}}{\mathbb{R}} = 0$. If S is the canonical spray again, then from (5.2b)

$$\forall X, Y \in \mathfrak{X}(\mathcal{T}M) : 0 = \overset{\mathrm{R}}{\mathbb{R}}(X,Y)S = \mathbb{R}(X,Y)S - \mathcal{C}(F\Omega(X,Y), S) =$$

$$\overset{3.8}{=} \mathbb{R}(X,Y)S \overset{(7.1)}{=} \overset{\circ}{\mathbb{R}}(X,Y)S \overset{(2.17)}{=} \Omega(X,Y),$$

so $\Omega = 0$ and, therefore, $\mathbb{R} = 0$. $\square$

7.2. Proposition. *A Finsler manifold (M, E) is a locally Minkowski manifold if and only if there exists a torsion-free, flat linear connection ∇ on M whose horizontal lift $\overset{h}{\nabla}$ is h-metrical with respect to the horizontal endomorphism arising from ∇.*

Proof.

Necessity. Assume (M, E) is a locally Minkowski manifold. Then (M, E) is a Berwald manifold as well; let ∇ be its linear connection (in the sense of 6.6(a)). Then ∇ is torsion-free and, by Proposition 7.1, it is flat. In view of Lemma 6.8, the horizontal lift $\overset{h}{\nabla}$ of ∇ is the Berwald connection of (M, E), which is h-metrical by Corollary 6.11.

Sufficiency. Suppose that ∇ is a torsion-free, flat linear connection on M, satisfying the condition

$$\forall X \in \mathfrak{X}(\mathcal{T}M) : \overset{h}{\nabla}_{hX} g_h = 0.$$

(h is the horizontal endomorphism arising from ∇). Clearly, the tension, the weak torsion and the curvature of h vanish. We claim that the Finsler connection $(\overset{h}{\nabla}, h)$ is of Berwald-type. To show this, by Theorem 4.3 it is enough to check (4.3a) and (4.3b). For each vector fields $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned} \mathbb{P}^1(X^h, Y^v) &:= v\mathbb{T}(X^h, Y^v) = v\left(\overset{h}{\nabla}_{X^h} Y^v - \overset{h}{\nabla}_{Y^v} X^h - [X^h, Y^v]\right) = \\ &\overset{(2.13)}{=} [X^h, Y^v] - [X^h, Y^v] = 0; \\ \mathbb{B}(X^h, Y^v) &:= h\mathbb{T}(X^h, Y^v) = 0; \end{aligned}$$

hence our statement. Let S be the geodesic spray of ∇ ([5], p. 173). Then $hS = S$ and for any vector field X on M

$$\overset{h}{\nabla}_{X^h} S = \overset{h}{\nabla}_{hX^h} hS \overset{(2.14d)}{=} hF[X^h, JS] = hF[X^h, C] = 0,$$

since the tension of h vanishes. Now we show that h is conservative. For each vector field X on M, we have

$$\begin{aligned} 0 &= \left(\overset{h}{\nabla}_{X^h} g_h\right)(S, S) = X^h g_h(S, S) - 2g_h\left(\overset{h}{\nabla}_{X^h} S, S\right) = X^h g_h(S, S) = \\ &\overset{(3.2b)}{=} X^h g(JS, JS) = X^h g(C, C) \overset{(3.3a)}{=} 2X^h E = 2d_h E(X^c), \end{aligned}$$

which gives the result. Finally we check that the $(h)h$-torsion $\mathbb{A}$ of $\overset{h}{\nabla}$ vanishes. For any two vector fields X, Y on M,

$$\begin{aligned} \mathbb{A}(X^c, Y^c) &= h\mathbb{T}(hX^c, hY^c) = h\mathbb{T}(X^h, Y^h) = h\left(\overset{h}{\nabla}_{X^h} Y^h - \overset{h}{\nabla}_{Y^h} X^h - [X^h, Y^h]\right) = \\ &\overset{(2.13)}{=} h\left((\nabla_X Y)^h - (\nabla_Y X)^h - [X^h, Y^h]\right) = (\nabla_X Y)^h - (\nabla_Y X)^h - [X, Y]^h = \\ &= (\nabla_X Y - \nabla_Y X - [X, Y])^h = 0, \end{aligned}$$

since ∇ is torsion-free. Now we infer from Theorem 4.3 that h is the Barthel endomorphism and, therefore, $(\overset{h}{\nabla}, h)$ is the Berwald connection. Hence (M, E) is a Berwald manifold satisfying the condition $\Omega = -\frac{1}{2}[h, h] = 0$; i.e., (M, E) is a locally Minkowski manifold. $\square$

7.3. Locally affine structures. Recall that an atlas $\mathcal{A} = \big(\mathcal{U}_\alpha, (u^i_\alpha)_{i=1}^n\big)_{\alpha\in A}$ of a manifold M is said to be a *locally affine structure* on M if the transition functions

$$\gamma_{\alpha\beta} : p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \mapsto \gamma_{\alpha\beta}(p) := \left(\frac{\partial u^i_\beta}{\partial u^j_\alpha}(p)\right) \in \mathrm{GL}(\mathbb{R}^n)$$

$$((\alpha,\beta) \in A \times A)$$

are constant. If $X, Y \in \mathfrak{X}(\mathcal{U}_\alpha)$, $Y = Y^i \frac{\partial}{\partial u^i_\alpha}$ and

$$\nabla^\alpha_X Y := X(Y^i)\frac{\partial}{\partial u^i_\alpha} \qquad (\alpha \in A),$$

then the family $(\nabla^\alpha)_{\alpha\in A}$ determines a well-defined linear connection ∇ on M. We say that ∇ *arises from the locally affine structure* $\mathcal{A}$. Clearly, the torsion tensor and the curvature tensor of ∇ vanish. Frobenius' classical theorem on integrable distributions assures that the converse is also true. Namely, we have

7.4. Lemma. *A linear connection on a manifold M which has zero curvature and torsion arises from a locally affine structure on M.*

This is proved e.g. in [5].

7.5. Theorem. *A Finsler manifold (M,E) is a locally Minkowski manifold if and only if there exists an atlas $\big(\mathcal{U}_\alpha, (u^i_\alpha)_{i=1}^n\big)_{\alpha\in A}$ on M such that in the induced atlas $\big(\pi^{-1}(\mathcal{U}_\alpha), (x^i_\alpha, y^i_\alpha)_{i=1}^n\big)_{\alpha\in A}$*

$$\forall\alpha \in A : \frac{\partial E}{\partial x^i_\alpha} = 0, \qquad 1 \le i \le n;$$

i.e. the energy function "does not depend on the position" over the induced charts.

Proof.

Necessity. Assume (M,E) is a locally Minkowski manifold. Then, in particular, (M,E) is a Berwald manifold and the curvature tensor of its (torsion-free) linear connection ∇ vanishes by Proposition 7.2. Applying Lemma 7.4 it follows that ∇ arises from a locally affine structure $\mathcal{A} = \big(\mathcal{U}_\alpha, (u^i_\alpha)_{i=1}^n\big)_{\alpha\in A}$ on M. Choose a chart $\big(\mathcal{U}, (u^i)_{i=1}^n\big) \in \mathcal{A}$ (the lower index α is omitted for simplicity). Then over $\mathcal{U}$

$$\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j} = 0, \qquad 1 \le i,j \le n.$$

Since the Berwald connection of (M,E) is just $(\overset{h}{\nabla}, h)$ by Lemma 6.8 (h is the Barthel endomorphism), we have on the one hand

$$\overset{\circ}{D}_{\left(\frac{\partial}{\partial u^i}\right)^h}\left(\frac{\partial}{\partial u^j}\right)^v = \overset{h}{\nabla}_{\left(\frac{\partial}{\partial u^i}\right)^h}\left(\frac{\partial}{\partial u^j}\right)^v \overset{(2.13)}{=} \left(\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j}\right)^v = 0 \qquad (1 \le i,j \le n).$$

On the other hand, for each indices $i, j \in \{1, \dots, n\}$

$$\overset{\circ}{D}_{\left(\frac{\partial}{\partial u^i}\right)^h}\left(\frac{\partial}{\partial u^j}\right)^v \overset{4.11}{=} \left[\left(\frac{\partial}{\partial u^i}\right)^h, \left(\frac{\partial}{\partial u^j}\right)^v\right] \overset{6.9(\mathrm{ii})}{=} G^k_{ij}\frac{\partial}{\partial y^k},$$

consequently
$$G^k_{ij} = 0, \qquad 1 \leq i, j, k \leq n.$$
Notice that the functions G^k_i (introduced in 6.9) are homogeneous of degree 1. Thus, by Euler's theorem,
$$G^k_i = y^j \frac{\partial G^k_i}{\partial y^j} \overset{(6.9c)}{=} y^j G^k_{ij} = 0, \qquad 1 \leq i, k \leq n.$$
Hence
$$\left(\frac{\partial}{\partial u^i}\right)^h = \frac{\partial}{\partial x^i}, \qquad 1 \leq i \leq n.$$
Since the Barthel endomorphism is conservative, i.e. $d_h E = 0$, we deduce
$$0 = d_h E\left(\frac{\partial}{\partial u^i}\right)^c = dE\left(\frac{\partial}{\partial u^i}\right)^h = dE\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial E}{\partial x^i}, \qquad 1 \leq i \leq n.$$
It means that the atlas $\mathcal{A}$ has the desired property.

Sufficiency. Assume the condition holds. Again, choose a chart $\big(\mathcal{U}, (u^i)_{i=1}^n\big) \in \mathcal{A}$. Then we have
$$\frac{\partial E}{\partial x^i} = 0, \qquad 1 \leq i \leq n.$$
Let h be the Barthel endomorphism. The coordinate expression of the property $d_h E = 0$ reduces to the relation
$$G^k_i \frac{\partial E}{\partial y^k} = 0, \qquad 1 \leq i \leq n.$$
Now from (6.9a) we deduce
$$G^k = 0, \qquad 1 \leq k \leq n.$$
Hence the functions G^k_i, G^k_{ij} and G^k_{ijl} also vanish over $\mathcal{U}$. The vanishing of the functions G^k_{ijl} means that $\overset{\circ}{\mathbb{P}} = 0$, therefore (M, E) is a Berwald manifold (6.9(iii)). Finally, the relations
$$G^k_i = 0, \quad G^k_{ij} = 0, \qquad (1 \leq i, j, k \leq n)$$
imply that $\Omega = 0$, and this ends the proof. □

7.6. Example. Let (M, E) be a Randers manifold with energy function $E = \frac{1}{2}L^2$; $L := L_\alpha + \widetilde{\beta}$. In view of Kikuchi's theorem (see [18]), (M, E) *is a locally Minkowski manifold if and only if the following two conditions are satisfied:*

(a) β is parallel with respect to the Lévi–Civita connection of the metric α;

(b) the Lévi–Civita connection has zero curvature.

(Both necessity and sufficiency can be seen from (6.6c) and Proposition 7.2.)

II. WAGNER CONNECTIONS AND WAGNER MANIFOLDS

1. Conformal equivalence of Riemann – Finsler metrics

1.1. Gradient operator on a Finsler manifold. Let (M, E) be a Finsler manifold with fundamental form ω. Consider a smooth function $\varphi : TM \to \mathbb{R}$. Since the fundamental form is nondegenerate, there exists a unique vector field $\operatorname{grad}\varphi \in \mathfrak{X}(\mathcal{T}M)$ such that

$$\imath_{\operatorname{grad}\varphi}\omega = d\varphi; \tag{1.1}$$

this vector field is called the *gradient* of φ.

1.2. Proposition. *Let (M, E) be a Finsler manifold and suppose that $\varphi \in C^\infty(TM)$ is a vertical lift: $\varphi = f \circ \pi$ $(f \in C^\infty(M))$. Then the gradient vector field of φ has the following properties:*

(1.2a) $\operatorname{grad}\varphi \in \mathfrak{X}^v(\mathcal{T}M)$;

(1.2b) $[C, \operatorname{grad}\varphi] = -\operatorname{grad}\varphi$;

(1.2c) $\operatorname{grad}\varphi(E) = f^c$;

(1.2d) $\imath_{F\operatorname{grad}\varphi}\mathcal{C} = -\dfrac{1}{2}[J, \operatorname{grad}\varphi]$ *(F is an almost complex structure associated with an arbitrary horizontal endomorphism)*;

(1.2e) *if* $\operatorname{grad}\varphi = \mu C$ $(\mu \in C^\infty(\mathcal{T}M))$ *then* $\mu = 0$ *and, consequently, the function* $f \in C^\infty(M)$ *is constant.*

Proof. For a proof of (1.2a)–(1.2c) and (1.2e) we refer to [35]. To verify (1.2d), let $Y, Z \in \mathfrak{X}(M)$ be arbitrary vector fields. Then, applying some well-known identities concerning the vertical and horizontal lifts of a vector field (see I.1.16) we get:

$$\begin{aligned}
2g(\mathcal{C}(F\operatorname{grad}\varphi, Y^c), Z^v) &\overset{\text{I.3.7}}{=} 2g\big(\mathcal{C}(Y^c, F\operatorname{grad}\varphi), Z^v\big) = \\
&= Y^v g(\operatorname{grad}\varphi, Z^v) - g\big(J[Y^v, F\operatorname{grad}\varphi], Z^v\big) = \\
&= Y^v\big((Zf)^v\big) - g\big([J, Y^v]F\operatorname{grad}\varphi, Z^v\big) - \\
&- g\big([Y^v, \operatorname{grad}\varphi], Z^v\big) = -g\big([Y^v, \operatorname{grad}\varphi], Z^v\big) = \\
&= -g\big([J, \operatorname{grad}\varphi]Y^c, Z^v\big);
\end{aligned}$$

hence our assertion. $\square$

1.3. Remarks. Hypothesis as in Proposition 1.2. An easy calculation shows that for any vector field $JX \in \mathfrak{X}^v(TM)$

$$D_{JX}\operatorname{grad}\varphi = -\mathcal{C}(F\operatorname{grad}\varphi, X) \overset{(1.2\mathrm{d})}{=} \frac{1}{2}[J, \operatorname{grad}\varphi]X, \tag{1.3a}$$

where the covariant derivative is taken with respect to the classical Cartan connection. On the other hand for any vector field $X \in \mathfrak{X}(M)$,

$$\begin{aligned} D_{\operatorname{grad}\varphi}X^v &\overset{\mathrm{I}.(4.5\mathrm{b})}{=} D_{X^v}\operatorname{grad}\varphi - [X^v, \operatorname{grad}\varphi] = \\ &\overset{(1.3\mathrm{a})}{=} -\mathcal{C}(F\operatorname{grad}\varphi, X^c) - [X^v, \operatorname{grad}\varphi] = \\ &= -\mathcal{C}(F\operatorname{grad}\varphi, X^c) - [J, \operatorname{grad}\varphi]X^c \overset{(1.2\mathrm{d})}{=} \\ &= \mathcal{C}(F\operatorname{grad}\varphi, X^c), \quad \text{i.e.} \end{aligned}$$

$$D_{\operatorname{grad}\varphi}X^v = \mathcal{C}(F\operatorname{grad}\varphi, X^c) \overset{(1.2\mathrm{d})}{=} -\frac{1}{2}[J, \operatorname{grad}\varphi]X^c. \tag{1.3b}$$

(Note that (1.3b) is *not* a tensorial relation!)

1.4. Definition. Consider the Finsler manifolds (M, E) and $(M, \widetilde{E})$ and let us denote by g and $\widetilde{g}$ their Riemann–Finsler metrics. g and $\widetilde{g}$ are said to be *conformally equivalent* if there exists a positive smooth function $\varphi : \mathcal{T}M \to \mathbb{R}$ such that

$$\widetilde{g} = \varphi g. \tag{1.4}$$

The function φ is called the *scale function* or the *proportionality function*. If the scale function is constant, then we say that the conformal change is *homothetic*.

1.5. Remark. If $\widetilde{g} = \varphi g$ then

$$\widetilde{E} \overset{\mathrm{I}.(3.3\mathrm{a})}{=} \frac{1}{2}\widetilde{g}(C, C) = \frac{1}{2}\varphi g(C, C) \overset{\mathrm{I}.(3.3\mathrm{a})}{=} \varphi E. \tag{1.5}$$

1.6. Lemma (Knebelman's observation)**.** *The scale function between conformally equivalent Riemann–Finsler metrics is a vertical lift, i.e., it can always be written in the form*

$$\varphi = \exp\circ\alpha^v := \exp\circ\alpha\circ\pi, \tag{1.6}$$

where $\alpha \in C^\infty(M)$*.*

Proof. First of all we show that the scale function is homogeneous of degree 0, i.e. $C\varphi = 0$. Using (1.5) we get that

$$\begin{aligned} C\widetilde{E} = C(\varphi E) = EC\varphi + \varphi CE \overset{\mathrm{I}.(3.1\mathrm{c})}{=} EC\varphi + \\ + 2\varphi E \overset{(1.5)}{=} EC\varphi + 2\widetilde{E} \overset{\mathrm{I}.(3.1\mathrm{c})}{=} EC\varphi + C\widetilde{E} \implies C\varphi = 0. \end{aligned}$$

Now for any vector fields $X, Y \in \mathfrak{X}(TM)$

$$
\begin{aligned}
\widetilde{g}(JX, JY) := \widetilde{\omega}(JX, Y) := dd_J\widetilde{E}(JX, Y) &\overset{(1.5)}{=} \\
&= dd_J(\varphi E)(JX, Y) = d(Ed_J\varphi + \varphi d_J E)(JX, Y) = \\
&= dE \wedge d_J\varphi(JX, Y) + Edd_J\varphi(JX, Y) + \\
&\quad + d\varphi \wedge d_J E(JX, Y) + \varphi dd_J E(JX, Y) \overset{\mathrm{I}.(1.8\mathrm{a})}{=} \\
&\quad = JX(E)JY(\varphi) + EJX(JY(\varphi)) - EJ[JX, Y](\varphi) + \\
&\quad + JX(\varphi)JY(E) + \varphi g(JX, JY).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 = &JX(E)JY(\varphi) + JX(\varphi)JY(E) + \\
&+ EJX\big(JY(\varphi)\big) - EJ[JX, Y](\varphi).
\end{aligned}
\tag{1.6a}
$$

By the substitution $Y := S_0$ (S_0 is an arbitrary semispray on M), we obtain:

$$
\begin{aligned}
0 = JX(E)C\varphi + JX(\varphi)CE + EJX(C\varphi) - EJ[JX, S_0](\varphi) &\overset{\mathrm{I}.1.15,\mathrm{I}.(3.1\mathrm{c})}{=} \\
= 2EJX(\varphi) - EJX(\varphi) = Ed_J\varphi(X) \implies d_J\varphi = 0.&
\end{aligned}
$$

According to I.1.12 this implies that φ is a vertical lift, which ends the proof. $\square$

1.7. Proposition. *If a Finsler manifold (M, E) with the Riemann – Finsler metric g and a function $\alpha \in C^\infty(M)$ are given, then $\widetilde{g} = \varphi g$ ($\varphi := \exp \circ \alpha^v$) is the Riemann – Finsler metric of the Finsler manifold $(M, \widetilde{E})$, where $\widetilde{E} := \varphi E$.*

Proof. It is enough to show that the form $\widetilde{\omega} := dd_J\widetilde{E}$ is nondegenerate. Since $\widetilde{E} := \varphi E$ we get immediately the relation

$$\widetilde{\omega} = d\varphi \wedge d_J E + \varphi\omega. \tag{1.7a}$$

Then the following assertions are equivalent:

$$0 = i_X\widetilde{\omega}; \tag{1.7b}$$

$$0 = X\varphi d_J E - JX(E)d\varphi + \varphi i_X\omega. \tag{1.7c}$$

Applying both sides of (1.7c) to an arbitrary vertical vector field $JY (Y \in \mathfrak{X}(TM))$ we obtain:

$$0 = \varphi i_X\omega(JY) := \varphi\omega(X, JY) = -\varphi g(JX, JY).$$

Therefore $JX = 0$ and thus $X \in \mathfrak{X}^v(TM)$. Hence

$$0 = i_X\widetilde{\omega} \iff 0 = \varphi i_X\omega \iff X = 0,$$

which means that $\widetilde{\omega}$ is nondegenerate .

Finally, for any vector fields $X, Y \in \mathfrak{X}(TM)$,

$$\widetilde{\omega}(JX, Y) \overset{(1.7\mathrm{a})}{=} \varphi\omega(JX, Y) = \varphi g(JX, JY) = \widetilde{g}(JX, JY),$$

i.e. the Riemann – Finsler metric of $(M, \widetilde{E})$ coincides with $\widetilde{g}$ as was to be stated. $\square$

1.8. Remark. According to Proposition 1.7 we also speak of a *conformal change* $\widetilde{g} = \varphi g$ $(\varphi := \exp \circ \alpha^v)$ *of the metric* g.

1.9. Proposition. *Let* (M, E) *and* $(M, \widetilde{E})$ *be Finsler manifolds with Riemann–Finsler metrics* g *and* $\widetilde{g}$*, respectively.* g *and* $\widetilde{g}$ *are conformally equivalent if and only if*

$$\frac{d_J E}{E} = \frac{d_J \widetilde{E}}{\widetilde{E}} \quad \text{(over } \mathcal{T}M\text{)}.$$

In particular, $\frac{d_J E}{E}$ *is invariant under the conformal changes of the Riemann–Finsler metric.*

Proof. The necessity is clear: if $\widetilde{g} = \varphi g$, then $\widetilde{E} = \varphi E$, where $d_J \varphi = 0$ by Lemma 1.6, and so

$$\frac{d_J \widetilde{E}}{\widetilde{E}} = \frac{d_J(\varphi E)}{\varphi E} = \frac{E d_J \varphi + \varphi d_J E}{\varphi E} = \frac{d_J E}{E}.$$

Suppose, conversely, that over $\mathcal{T}M$

$$\frac{d_J E}{E} = \frac{d_J \widetilde{E}}{\widetilde{E}}.$$

Then we get immediately the relation

$$d_J(\ln \circ E) = d_J(\ln \circ \widetilde{E})$$

or, equivalently,

$$d_J\left(\ln \circ \frac{\widetilde{E}}{E}\right) = 0.$$

This means by I.1.12 that the function $\ln \circ \frac{\widetilde{E}}{E}$ and, therefore, $\frac{\widetilde{E}}{E}$ is a vertical lift. So there is a positive function $f \in C^\infty(M)$ such that $\frac{\widetilde{E}}{E} = f \circ \pi$. If $\varphi := f \circ \pi$, then we have for any vector fields $X, Y \in \mathfrak{X}(\mathcal{T}M)$ that

$$\begin{aligned} \widetilde{g}(JX, JY) := \widetilde{\omega}(JX, Y) &\overset{(1.7.a)}{=} d\varphi \wedge d_J E(JX, Y) + \\ &+ \varphi\omega(JX, Y) = \varphi\omega(JX, Y) = \\ &= \varphi g(JX, JY); \end{aligned}$$

hence

$$\widetilde{g} = \varphi g,$$

as was to be shown. $\square$

1.10. Proposition. *The Cartan tensor is invariant under a conformal change of a Riemann–Finsler metric, while the lowered Cartan tensor changes by the formula*

$$\widetilde{\mathcal{C}}_\flat = \varphi \mathcal{C}_\flat,$$

where φ *is the scale function.*

Proof. Consider the conformal change $\widetilde{g} = \varphi g$. Then for any vector fields $X, Y, Z \in \mathfrak{X}(TM)$ we obtain that

$$\begin{aligned}
2\widetilde{g}(\widetilde{\mathcal{C}}(X,Y), JZ) &\overset{\mathrm{I.(3.4a)}}{=} \mathcal{L}_{JX}\left(J^*\widetilde{g}_{\widetilde{h}}\right)(Y,Z) = \\
&= JX\widetilde{g}_{\widetilde{h}}(JY,JZ) - \widetilde{g}_{\widetilde{h}}(J[JX,Y],JZ) - \\
&- \widetilde{g}_{\widetilde{h}}\big(JY, J[JX,Z]\big) = JX\widetilde{g}(JY,JZ) - \\
&- \widetilde{g}\big(J[JX,Y],JZ\big) - \widetilde{g}\big(JY, J[JX,Z]\big) = \\
&= JX\big(\varphi g(JY,JZ)\big) - \varphi\Big(g\big(J[JX,Y],JZ\big) + \\
&+ g\big(JY, J[JX,Z]\big)\Big) \overset{\text{Lemma 1.6}}{=} \\
&= \varphi\Big(JXg(JY,JZ) - g\big(J[JX,Y],JZ\big) - g\big(JY,J[JX,Z]\big)\Big) \\
&= 2\varphi g\big(\mathcal{C}(X,Y),JZ\big) = 2\widetilde{g}\big(\mathcal{C}(X,Y),JZ\big),
\end{aligned}$$

therefore $\widetilde{\mathcal{C}} = \mathcal{C}$. This relation implies immediately that $\widetilde{\mathcal{C}}_\flat = \varphi\mathcal{C}_\flat$. $\square$

1.11. Remark. It is well-known that the vanishing of the conformal invariant tensor $\mathcal{C}$ implies that the Finsler manifold (M,E) is Riemannian.

1.12. Proposition. *Let (M,E) be a Finsler manifold and $\beta \in C^\infty(M)$. Then the tensors*

$$E[J, \operatorname{grad}\beta^v], \qquad d_J E \otimes \operatorname{grad}\beta^v$$

are invariant under any conformal change of the metric g.

Proof. Let us consider the conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$). Since $\operatorname{grad}\beta^v \in \mathfrak{X}^v(\mathcal{T}M)$ (see Proposition 1.2), we get from (1.7a) the relation

$$\imath_{\operatorname{grad}\beta^v}\widetilde{\omega} = \varphi\imath_{\operatorname{grad}\beta^v}\omega = \varphi d\beta^v \implies \frac{1}{\varphi}\operatorname{grad}\beta^v = \widetilde{\operatorname{grad}}\beta^v.$$

Therefore

$$\widetilde{E}[J, \widetilde{\operatorname{grad}}\beta^v] = \varphi E\big[J, \frac{1}{\varphi}\operatorname{grad}\beta^v\big] = E[J, \operatorname{grad}\beta^v],$$

since φ is a vertical lift.

In the same way

$$d_J\widetilde{E} \otimes \widetilde{\operatorname{grad}}\beta^v = d_J(\varphi E) \otimes \frac{1}{\varphi}\operatorname{grad}\beta^v = \varphi d_J E \otimes \frac{1}{\varphi}\operatorname{grad}\beta^v = d_J E \otimes \operatorname{grad}\beta^v. \quad \square$$

2. Some transformation formulas

2.1. Theorem. *Suppose that g and $\widetilde{g}$ are conformally equivalent Riemann – Finsler metrics on M, namely*

$$\widetilde{g} = \varphi g; \quad \varphi = \exp \circ \alpha \circ \pi, \quad \alpha \in C^\infty(M).$$

Then the corresponding canonical sprays satisfy the relation

$$\widetilde{S} = S - \alpha^c C + E \operatorname{grad} \alpha^v. \tag{2.1}$$

Proof. On the one hand, since $\widetilde{E} = \varphi E$, we have

$$-d\widetilde{E} = -d(\varphi E) = -\varphi dE - E d\varphi = i_{\varphi S}\omega - E d\varphi.$$

On the other hand,

$$\begin{aligned} -d\widetilde{E} &= i_{\widetilde{S}}\widetilde{\omega} = i_{\widetilde{S}} dd_J(\varphi E) = i_{\widetilde{S}} d(E d_J \varphi + \varphi d_J E) = \\ &= i_{\widetilde{S}}(d\varphi \wedge d_J E + \varphi dd_J E) = (i_{\widetilde{S}} d\varphi) d_J E - \\ &- (i_{\widetilde{S}} d_J E) d\varphi + \varphi i_{\widetilde{S}}\omega = \widetilde{S}(\varphi) d_J E - dE(J\widetilde{S}) d\varphi + \\ &+ \varphi i_{\widetilde{S}}\omega \overset{\mathrm{I.}(1.11\mathrm{b}),\mathrm{I.}(3.3\mathrm{c});(1.2\mathrm{c})}{=} i_{\operatorname{grad}\varphi(E)C}\omega - 2E d\varphi + i_{\varphi\widetilde{S}}\omega. \end{aligned}$$

Comparing the right sides of these relations, we obtain that

$$i_{\varphi\widetilde{S}}\omega = i_{\varphi S}\omega - i_{\operatorname{grad}\varphi(E)C}\omega + E d\varphi = i_{\varphi S - \operatorname{grad}\varphi(E)C + E\operatorname{grad}\varphi}\omega.$$

Hence

$$\widetilde{S} = S - \frac{1}{\varphi}\operatorname{grad}\varphi(E) C + E\frac{1}{\varphi}\operatorname{grad}\varphi.$$

Since

$$\operatorname{grad}\varphi = \operatorname{grad}(\exp \circ \alpha^v) = (\exp' \circ \alpha^v) \operatorname{grad}\alpha^v = \varphi \operatorname{grad}\alpha^v$$

and therefore

$$\frac{1}{\varphi}\operatorname{grad}\varphi(E) = \operatorname{grad}\alpha^v(E) \overset{(1.2\mathrm{c})}{=} \alpha^c,$$

(2.1) is proved. □

2.2. Corollary. *Under the conditions of Theorem 2.1, the Barthel endomorphisms are related as follows:*

$$\widetilde{h} = h - \frac{1}{2}(\alpha^c J + d\alpha^v \otimes C) + \frac{1}{2}E[J, \operatorname{grad}\alpha^v] + \frac{1}{2} d_J E \otimes \operatorname{grad}\alpha^v. \tag{2.2}$$

Proof. $\widetilde{h} := \frac{1}{2}\big(1_{\mathfrak{X}(TM)} + [J, \widetilde{S}]\big) \overset{(2.1)}{=} h - \frac{1}{2}[J, \alpha^c C] + \frac{1}{2}[J, E \operatorname{grad}\alpha^v]$. Here

$$\begin{aligned} -[J, \alpha^c C] &= [\alpha^c C, J] \overset{\mathrm{I.}(1.6\mathrm{i})}{=} \alpha^c [C, J] + d\alpha^c \wedge i_C J - \\ &- d_J \alpha^c \otimes C \overset{\mathrm{I.}(1.8\mathrm{d})}{=} -\alpha^c J - d\alpha^v \otimes C, \end{aligned} \tag{2.2a}$$

$$\begin{aligned} [J, E\operatorname{grad}\alpha^v] &\overset{\mathrm{I.}(1.6\mathrm{i})}{=} E[J, \operatorname{grad}\alpha^v] - dE \wedge i_{\operatorname{grad}\alpha^v} J + d_J E \otimes \operatorname{grad}\alpha^v \\ &= E[J, \operatorname{grad}\alpha^v] + d_J E \otimes \operatorname{grad}\alpha^v, \end{aligned} \tag{2.2b}$$

thus we obtain the desired relation. □

2.3. Definition. Suppose that $L \in \Psi^{\ell}(TM)$ is semibasic (cf. I.1.9.). The *semibasic trace* $\widetilde{L}$ of L is the semibasic scalar $(\ell-1)$-form defined by recurrence as follows:

(2.3a) if $\ell = 1$, then $\widetilde{L} := \operatorname{trace}(F \circ L)$, where F is the associated almost complex structure of an arbitrarily chosen horizontal endomorphism;

(2.3b) if $\ell > 1$, then $\imath_X \widetilde{L} := \widetilde{\imath_X L}$ for all $X \in \mathfrak{X}(TM)$.

2.4. Lemma. *For any $\eta \in \Omega^k(TM)$ semibasic form:*

$$\widetilde{\eta \otimes C} = (-1)^{k+1} \imath_S \eta, \tag{2.4a}$$

where S is an arbitrary semispray on M.

$$\widetilde{J} = n. \tag{2.4b}$$

For a proof see [35], p. 173 and [43].

2.5. Divergence operator on a Finsler manifold. Let (M, E) be a Finsler manifold and consider the volume form

$$w := \frac{(-1)^{n(n+1)/2}}{n!} \omega^n \tag{2.5a}$$

on $\mathcal{T}M$. The *divergence* of a vector field $X \in \mathfrak{X}(TM)$ is the function δX given by

$$(\delta X) w = \mathcal{L}_X w. \tag{2.5b}$$

We have the following formulas (see [12] and [35]):

$$\forall\, X \in \mathfrak{X}(TM) : \delta(JX) = \widetilde{[J, JX]} + 2\widetilde{\mathcal{C}}(X); \tag{2.5c}$$

$$\delta C = n; \tag{2.5d}$$

$$\forall\, \varphi \in C^{\infty}(TM) : \delta \operatorname{grad} \varphi = 0. \tag{2.5e}$$

2.6. Application. A well-known, classical result states (in H. Weyl's terminology) that "*the projective and conformal properties of a Finsler space determine its metric properties uniquely*" [30], p. 226. Now we are going to formulate this nice and important theorem in our framework and prove it *purely intrinsically*.

Definition. (cf. [6]) (a) Two sprays S and $\widetilde{S}$ given on the manifold M are said to be *projectively equivalent* if there is a function $\lambda : TM \to \mathbb{R}$ satisfying the following conditions:

(2.6a) λ is smooth on $\mathcal{T}M$, C^0 on TM.

$$\widetilde{S} = S + \lambda C. \tag{2.6b}$$

(b) We say that the Finsler manifolds (M, E) and $(M, \widetilde{E})$ are projectively equivalent if their canonical sprays have the property described in (a).

Remark. (2.6a) and (2.6b) easily imply that λ is homogeneous of degree 1.

Theorem. *Suppose that (M, E) and $(M, \widetilde{E})$ are projectively equivalent Finsler manifolds of dimension $n > 1$. If their Riemann–Finsler metrics are conformally equivalent as well then the conformal change is homothetic.*

Proof. Keeping the previous notation, now we have by (2.6b) and (2.1) the relations

$$\widetilde{S} = S + \lambda C \qquad \text{and} \qquad \widetilde{S} = S - \alpha^c C + E \operatorname{grad} \alpha^v$$

simultaneously. From these it follows that

$$\lambda C = -\alpha^c C + E \operatorname{grad} \alpha^v. \tag{2.6c}$$

Applying both sides to the energy function E, we obtain by I.(3.1c) and (1.2c) that

$$2\lambda E = -2\alpha^c E + \alpha^c E,$$

i.e.

$$\lambda = -\frac{1}{2}\alpha^c.$$

Substituting this into (2.6c), we get the relation

$$\operatorname{grad} \alpha^v = \frac{\alpha^c}{2E} C.$$

Let $\mu := \frac{\alpha^c}{2E}$. In view of I.(1.11a) and I.(3.1c), μ is homogeneous of degree -1. Thus we have:

$$\operatorname{grad} \alpha^v = \mu C, \tag{2.6d}$$

$$C\mu = -\mu. \tag{2.6e}$$

Let S_0 be an arbitrary semispray on M. First we note that

$$\tilde{\mathcal{C}}(S_0) = 0. \tag{2.6f}$$

Indeed,

$$\begin{aligned} n &\overset{(2.5d)}{=} \delta C = \delta(JS_0) \overset{(2.5c)}{=} \widetilde{[J, JS_0]} + 2\tilde{\mathcal{C}}(S_0) = \\ &= \widetilde{[J, C]} + 2\tilde{\mathcal{C}}(S_0) \overset{\mathrm{I.}(1.8d)}{=} \tilde{J} + 2\tilde{\mathcal{C}}(S_0) \overset{\text{Lemma 2.4}}{=} \\ &= n + 2\tilde{\mathcal{C}}(S_0), \end{aligned}$$

so (2.6f) is true. Secondly, we claim that

$$\widetilde{[J, \operatorname{grad} \alpha^v]} = 0. \tag{2.6g}$$

To see this, consider the vector field $X := \mu S_0$. Then

$$JX = \mu J S_0 = \mu C \overset{(2.6d)}{=} \operatorname{grad} \alpha^v,$$

therefore

$$0 \overset{(2.5e)}{=} \delta \operatorname{grad} \alpha^v = \delta(JX) \overset{(2.5c)}{=} \widetilde{[J, JX]} + 2\tilde{\mathcal{C}}(X) =$$

$$= \widetilde{[J, \operatorname{grad} \alpha^v]} + 2\mu\tilde{\mathcal{C}}(S_0) \overset{(2.6f)}{=} \widetilde{[J, \operatorname{grad} \alpha^v]},$$

so (2.6g) is also true. Now we come back to (2.6d). Taking Frölicher-Nijenhuis bracket with J we get the relation

$$[J, \operatorname{grad} \alpha^v] = [J, \mu C]. \tag{2.6h}$$

We calculate the semibasic trace of both sides.

$$0 \overset{(2.6g)}{=} \widetilde{[J, \operatorname{grad} \alpha^v]} \overset{(2.6h)}{=} \widetilde{[J, \mu C]} \overset{\mathrm{I}.(1.6i)}{=} \widetilde{\mu J} -$$

$$- \widetilde{d\mu \wedge i_C J} + \widetilde{d_J \mu \otimes C} = \mu \tilde{J} + \widetilde{d_J \mu \otimes C} =$$

$$\overset{(2.4a),(2.4b)}{=} \mu n + i_{S_0} d_J \mu = \mu n + i_{S_0} i_J d\mu =$$

$$= \mu n + d\mu(JS_0) = \mu n + C\mu \overset{(2.6e)}{=} \mu n - \mu = (n-1)\mu,$$

so $\mu = 0$. This implies in view of (2.6d) that $\operatorname{grad} \alpha^v = 0$. Thus

$$d\alpha^v = i_{\operatorname{grad} \alpha^v} \omega = 0,$$

therefore $d\alpha = 0$. Because M is connected (cf. I.1.1), this means that the function α is constant which was to be proved. $\square$

Remark. Observe that under the hypothesis of the Theorem the projective equivalence is trivial: the function λ vanishes.

Corollary. *Suppose that the Riemann – Finsler metrics g and $\widetilde{g}$ on M are conformally equivalent:*

$$\widetilde{g} = \varphi g; \quad \varphi = \exp \circ \alpha \circ \pi, \quad \alpha \in C^\infty(M).$$

Then the following assertions are equivalent:

(i) *The conformal change is homothetic.*
(ii) *The canonical sprays S and $\widetilde{S}$ coincide.*
(iii) *The Barthel endomorphisms h and $\widetilde{h}$ coincide.*

Proof. (i) $\Longrightarrow$ (ii) Indeed, if (i) holds, then α^c and $\operatorname{grad} \alpha^v$ vanish so (2.1) implies that $S = \widetilde{S}$.

(ii) $\Longrightarrow$ (i) This follows immediately from the Theorem.
(ii) $\Longrightarrow$ (iii) This is evident.
(iii) $\Longrightarrow$ (ii) From the hypothesis that (iii) holds we obtain the relation

$$[J, \widetilde{S}] = [J, S].$$

Applying both sides to the spray S, we have by I.(1.4a) that

$$[C, \widetilde{S}] - J[S, \widetilde{S}] = [C, S] - J[S, S].$$

Using I.(1.10d), it follows that

$$\widetilde{S} - J[S, \widetilde{S}] = S.$$

Let $Z := S - \widetilde{S}$. Then $Z \in \mathfrak{X}^v(\mathcal{T}M)$ and

$$J[S, \widetilde{S}] = J[Z, \widetilde{S}].$$

Here, by I.(1.15),

$$J[Z, \widetilde{S}] = Z = S - \widetilde{S}.$$

Therefore

$$\widetilde{S} - S + \widetilde{S} = S$$

hence $\widetilde{S} = S$, proving the implication (iii) $\Rightarrow$ (ii). $\square$

2.7. Remarks. Lots of further transformation formulas and applications can be found in Hashiguchi's fundamental work [14]. The author surveis them systematically including the most complicated ones as well, i.e., the relations between curvatures of notable Finsler connections. In order to illustrate the problem we are going to derive how the second Cartan tensors are related in case of conformally equivalent Riemann – Finsler metrics. Notations as usual.

2.8. Proposition. *The second Cartan tensor associated with the Barthel endomorphism h changes by the formula*

$$\begin{aligned}(2.8)\qquad \widetilde{\mathcal{C}}'(X,Y) = \mathcal{C}'(X,Y) - \frac{1}{2}\Big(&\alpha^c\mathcal{C}(X,Y) + \\ &+ JX(E)\mathcal{C}(F\operatorname{grad}\alpha^v, Y) + JY(E)\mathcal{C}(F\operatorname{grad}\alpha^v, X) + \\ &+ \mathcal{C}_\flat(F\operatorname{grad}\alpha^v, X, Y)C\Big) - E(D_{\operatorname{grad}\alpha^v}\mathcal{C})(X,Y),\end{aligned}$$

where $\varphi := \exp\circ\alpha^v$ is the scale function.

Proof. It is obvious from (1.2d) and Corollary 2.2 that for any vector fields X, Y, $Z \in \mathfrak{X}(M)$:

$$\begin{aligned}(2.8a)\qquad X^{\widetilde{h}} = X^h - \frac{1}{2}\alpha^c X^v - \frac{1}{2}(X\alpha)^v C - E\mathcal{C}(F\operatorname{grad}\alpha^v, X^c) + \\ + \frac{1}{2}X^v(E)\operatorname{grad}\alpha^v\end{aligned}$$

and, consequently,

$$\begin{aligned}(2.8b)\qquad [X^{\widetilde{h}}, Y^v] = [X^h, Y^v] + \frac{1}{2}\left((Y\alpha)^v X^v + (X\alpha)^v Y^v\right) - \\ - E\left[\mathcal{C}(F\operatorname{grad}\alpha^v, X^c), Y^v\right] + Y^v(E)\mathcal{C}(F\operatorname{grad}\alpha^v, X^c) + \\ + X^v(E)\mathcal{C}(F\operatorname{grad}\alpha^v, Y^c) - \frac{1}{2}g(X^v, Y^v)\operatorname{grad}\alpha^v.\end{aligned}$$

Therefore

$$2\widetilde{g}\big(\widetilde{\mathcal{C}}'(X^c,Y^c),Z^v\big) := X^{\widetilde{h}}\widetilde{g}(Y^v,Z^v) - \widetilde{g}\Big(\big[X^{\widetilde{h}},Y^v\big],Z^v\Big) - \\ - \widetilde{g}\Big(Y^v,\big[X^{\widetilde{h}},Z^v\big]\Big) = X^{\widetilde{h}}(\varphi)g(Y^v,Z^v) + \\ + \varphi\left(X^{\widetilde{h}}g(Y^v,Z^v) - g\Big(\big[X^{\widetilde{h}},Y^v\big],Z^v\Big) - g\Big(Y^v,\big[X^{\widetilde{h}},Z^v\big]\Big)\right).$$

Here

$$X^{\widetilde{h}}(\varphi) = X^{\widetilde{h}}(\exp\circ\alpha^v) = (\exp'\circ\alpha^v)X^{\widetilde{h}}(\alpha^v) = (\exp\circ\alpha^v)(X\alpha)^v = \varphi(X\alpha)^v,$$

$$2\widetilde{g}\big(\widetilde{\mathcal{C}}'(X^c,Y^c),Z^v\big) = 2\varphi g\big(\widetilde{\mathcal{C}}'(X^c,Y^c),Z^v\big).$$

Thus we obtain the formula

$$\begin{aligned} 2g\big(\widetilde{\mathcal{C}}'(X^c,Y^c),Z^v\big) &= (X\alpha)^v g(Y^v,Z^v) + X^{\widetilde{h}}g(Y^v,Z^v) - \\ &- g\Big(\big[X^{\widetilde{h}},Y^v\big],Z^v\Big) - g\Big(Y^v,\big[X^{\widetilde{h}},Z^v\big]\Big) \overset{(2.8a),(2.8b)}{=} \\ &= 2g(\mathcal{C}'(X^c,Y^c),Z^v) - \frac{1}{2}\alpha^c X^v g(Y^v,Z^v) - \\ &- E\mathcal{C}(F\operatorname{grad}\alpha^v,X^c)g(Y^v,Z^v) + \\ &+ Eg\big([\mathcal{C}(F\operatorname{grad}\alpha^v,X^c),Y^v],Z^v\big) + \\ &+ Eg\big(Y^v,[\mathcal{C}(F\operatorname{grad}\alpha^v,X^c),Z^v]\big) - \\ &- X^v(E)\cdot g\big(\mathcal{C}(F\operatorname{grad}\alpha^v,Y^c),Z^v\big) - \\ &- Y^v(E)g\big(\mathcal{C}(F\operatorname{grad}\alpha^v,X^c),Z^v\big) - \\ &- Z^v(E)g\big(\mathcal{C}(F\operatorname{grad}\alpha^v,X^c),Y^v\big). \end{aligned} \tag{2.8c}$$

Since the Cartan connection is metrical and its $(v)v$-torsion vanishes it follows that

$$\begin{aligned} &\mathcal{C}(F\operatorname{grad}\alpha^v,X^c)g(Y^v,Z^v) - g\big([\mathcal{C}(F\operatorname{grad}\alpha^v,X^c),Y^v],Z^v\big) - \\ &- g\big(Y^v,[\mathcal{C}(F\operatorname{grad}\alpha^v,X^c),Z^v]\big) = \\ &= g\big(D_{Y^v}\mathcal{C}(F\operatorname{grad}\alpha^v,X^c),Z^v\big) + g\big(D_{Z^v}\mathcal{C}(F\operatorname{grad}\alpha^v,X^c),Y^v\big) = \\ &\overset{\mathrm{I}.(5.4a)}{=} 2g\big((D_{\operatorname{grad}\alpha^v}\mathcal{C})(X^c,Y^c),Z^v\big) + \\ &+ g\big(\mathcal{C}(FD_{Y^v}\operatorname{grad}\alpha^v,X^c),Z^v\big) + g\big(\mathcal{C}(F\operatorname{grad}\alpha^v,FD_{Y^v}X^v),Z^v\big) + \\ &+ g\big(\mathcal{C}(FD_{Z^v}\operatorname{grad}\alpha^v,X^c),Y^v\big) + g\big(\mathcal{C}(F\operatorname{grad}\alpha^v,FD_{Z^v}X^v),Y^v\big). \end{aligned}$$

For example, using the symmetry properties of the (lowered) first Cartan tensor:

$$g\big(\mathcal{C}(FD_{Y^v}\operatorname{grad}\alpha^v,X^c),Z^v\big) + g\big(\mathcal{C}(F\operatorname{grad}\alpha^v,FD_{Z^v}X^v),Y^v\big) =$$

$$= g\big(\mathcal{C}(X^c, Z^c), D_{Y^v} \operatorname{grad} \alpha^v\big) + g\big(\mathcal{C}(F \operatorname{grad} \alpha^v, Y^c), D_{Z^v} X^v\big) =$$

$$\overset{\text{I.4.11}}{=} g\big(\mathcal{C}(X^c, Z^c), D_{Y^v} \operatorname{grad} \alpha^v\big) +$$

$$+ g\big(\mathcal{C}(F \operatorname{grad} \alpha^v, Y^c), \mathcal{C}(Z^c, X^c)\big) \overset{(1.3\text{a})}{=} 0.$$

Moreover,

$$g\big(\mathcal{C}(X^c, Y^c), Z^v\big) = \frac{1}{2} X^v g(Y^v, Z^v), \qquad Z^v(E) = g(C, Z^v).$$

By the substitution these results in (2.8c) it can be easily seen that

$$\begin{aligned} \widetilde{\mathcal{C}}'(X^c, Y^c) = \mathcal{C}'(X^c, Y^c) - \frac{1}{2}\Big(&\alpha^c \mathcal{C}(X^c, Y^c) + \\ &+ X^v(E)\mathcal{C}(F \operatorname{grad} \alpha^v, Y^c) + Y^v(E)\mathcal{C}(F \operatorname{grad} \alpha^v, X^c) + \\ &+ \mathcal{C}_\flat(F \operatorname{grad} \alpha^v, X^c, Y^c) C\Big) - E(D_{\operatorname{grad} \alpha^v} \mathcal{C})(X^c, Y^c), \end{aligned}$$

as was to be proved. □

2.9. Remarks. Following Hashiguchi's idea we can introduce a semibasic tensor $\mathbb{V} : \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \times \mathfrak{X}(\mathcal{T}M) \to \mathfrak{X}(\mathcal{T}M)$ as follows

$$\begin{aligned} \text{(2.9)} \qquad \mathbb{V}(Z, X, Y) := \frac{1}{2}\Big(&d_J E \otimes \mathcal{C}(Z, X, Y) + d_J E \otimes \mathcal{C}(X, Z, Y) + \\ &+ d_J E \otimes \mathcal{C}(Y, Z, X) + \mathcal{C}_\flat(Z, X, Y) C\Big) + E\big(D_{JZ}\mathcal{C}\big)(X, Y). \end{aligned}$$

Then (2.8) can be written in the form:

$$\widetilde{\mathcal{C}}' = \mathcal{C}' - \imath_{F \operatorname{grad} \alpha^v} \mathbb{V}.$$

It is clear that the tensor $\mathbb{V}$ is "semi"-invariant under a conformal change of the metrics, i.e.

$$\widetilde{\mathbb{V}} = \varphi \mathbb{V}.$$

Moreover (cf. [14], Proposition 4.1 and 4.2, p. 44), the following conditions are equivalent:

(a) $\mathbb{V} = 0$.
(b) $\mathcal{C}'$ is invariant under a conformal change of the metric.
(c) The hv-curvature $\mathbb{P}$ of the Cartan connection is invariant under a conformal change of the metric.

3. Wagner connections on a Finsler manifold

3.1. Definition. Let (M,E) be a Finsler manifold. The triplet $(\overline{D},\overline{h},\alpha)$ is said to be a *Wagner connection* on M if it satisfies the following conditions:

(3.1a) $(\overline{D},\overline{h})$ is a Finsler connection on $M, \alpha \in C^\infty(M)$;

(3.1b) $\overline{D}$ is metrical with respect to $g_{\overline{h}} : \overline{D}g_{\overline{h}} = 0$;

(3.1c) the $(v)v$-torsion $\overline{\mathbb{S}}^1$ of $\overline{D}$ vanishes: $\overline{\mathbb{S}}^1 = 0$;

(3.1d) $\overline{D}$ is $(h)h$-semisymmetric, i.e. the $(h)h$-torsion $\overline{\mathbb{A}}$ of $\overline{D}$ has the following form:

$$\overline{\mathbb{A}} = d\alpha^v \otimes \overline{h} - \overline{h} \otimes d\alpha^v;$$

(3.1e) the h-deflection $\overline{h}^*(DC)$ vanishes: $\overline{h}^*(DC) = 0$.

Then $\overline{h}$ is called a *Wagner endomorphism* on M.

3.2. Proposition. *Any Wagner endomorphism is a conservative horizontal endomorphism, i.e.* $d_{\overline{h}}E = 0$.

Proof. $\forall X \in (TM)$:

$$2d_{\overline{h}}E(X) \overset{\mathrm{I.(1.6e)}}{=} 2\overline{h}(X)E \overset{\mathrm{I.(3.3a)}}{=} \overline{h}(X)g(C,C) \overset{(3.1b)}{=} g\left(\overline{D}_{\overline{h}X}C, C\right) + g\left(C, \overline{D}_{\overline{h}X}C\right) \overset{(3.1e)}{=} 0.$$

□

3.3. Theorem. *The Wagner endomorphism $\overline{h}$ and the Barthel endomorphism h of a Finsler manifold are related as follows:*

$$\overline{h} = h + \alpha^c J - E[J, \operatorname{grad}\alpha^v] - d_J E \otimes \operatorname{grad}\alpha^v. \tag{3.3a}$$

Proof. Due to the 2nd local basis property, we can restrict ourselves to vertically and horizontally lifted vector fields, so let $X, Y, Z \in \mathfrak{X}(M)$ be arbitrary. From (3.1b) we get:

$$\begin{cases} X^{\overline{h}}g(Y^v,Z^v) = g\left(\overline{D}_{X^{\overline{h}}}Y^v, Z^v\right) + g\left(Y^v, \overline{D}_{X^{\overline{h}}}Z^v\right), \\ Y^{\overline{h}}g(Z^v,X^v) = g\left(\overline{D}_{Y^{\overline{h}}}Z^v, X^v\right) + g\left(Z^v, \overline{D}_{Y^{\overline{h}}}X^v\right), \\ -Z^{\overline{h}}g(X^v,Y^v) = -g\left(\overline{D}_{Z^{\overline{h}}}X^v, Y^v\right) - g\left(X^v, \overline{D}_{Z^{\overline{h}}}Y^v\right). \end{cases} \tag{3.3b}$$

Adding now both sides of (3.3b) it follows that

$$
\begin{aligned}
2g\left(\overline{D}_{X^{\overline{h}}}Y^v, Z^v\right) &= X^{\overline{h}}g(Y^v,Z^v) + Y^{\overline{h}}g(Z^v,X^v) - Z^{\overline{h}}g(X^v,Y^v) + \\
&+ g\left(X^v, \overline{D}_{Z^{\overline{h}}}Y^v - \overline{D}_{Y^{\overline{h}}}Z^v\right) + g\left(Y^v, \overline{D}_{Z^{\overline{h}}}X^v - D_{X^{\overline{h}}}Z^v\right) + \\
&+ g\left(Z^v, \overline{D}_{X^{\overline{h}}}Y^v - \overline{D}_{Y^{\overline{h}}}X^v\right) = X^{\overline{h}}g(Y^v,Z^v) + Y^{\overline{h}}g(Z^v,X^v) - \\
&- Z^{\overline{h}}g(X^v,Y^v) + g\left(X^v, \overline{F}\,\overline{\mathbb{A}}\left(Y^{\overline{h}}, Z^{\overline{h}}\right)\right) + g\left(Y^v, \overline{F}\,\overline{\mathbb{A}}\left(X^{\overline{h}}, Z^{\overline{h}}\right)\right) + \\
&+ g\left(Z^v, \overline{F}\,\overline{\mathbb{A}}\left(Y^{\overline{h}}, X^{\overline{h}}\right)\right) - g(X^v,[Y,Z]^v) - g(Y^v,[X,Z]^v) - \\
&- g(Z^v,[Y,X]^v) \overset{(3.1d)}{=} X^{\overline{h}}g(Y^v,Z^v) + Y^{\overline{h}}g(Z^v,X^v) - Z^{\overline{h}}g(X^v,Y^v) + \\
&+ 2g(X^v,Y^v)g(\operatorname{grad}\alpha^v, Z^v) - 2g(\operatorname{grad}\alpha^v, Y^v)g(X^v,Z^v) - \\
&- g(X^v,[Y,Z]^v) - g(Y^v,[X,Z]^v) - g(Z^v,[Y,X]^v).
\end{aligned}
$$

Applying an analogous "Christoffel process" to the Cartan connection (D,h) we get:

$$
\begin{aligned}
g\left(\overline{D}_{X^{\overline{h}}}Y^v - D_{X^h}Y^v, Z^v\right) &= g\left(\mathcal{C}(Y^c,Z^c), X^{\overline{h}} - X^h\right) + \\
&+ g\left(\mathcal{C}(X^c,Z^c), Y^{\overline{h}} - Y^h\right) - g\left(\mathcal{C}(X^c,Y^c), Z^{\overline{h}} - Z^h\right) + \\
&+ g(X^v,Y^v)g(\operatorname{grad}\alpha^v, Z^v) - g(\operatorname{grad}\alpha^v, Y^v)g(X^v,Z^v).
\end{aligned}
$$

From this follows that $\forall X, Y, Z \in \mathfrak{X}(TM)$:

$$
\begin{aligned}
\text{(3.3c)}\qquad g\left(\overline{D}_{\overline{h}X}JY - D_{\overline{h}X}JY, JZ\right) &= g\left(\mathcal{C}(X,Z), (\overline{h}-h)Y\right) - \\
&- g\left(\mathcal{C}(X,Y), \left(\overline{h}-h\right)Z\right) + g(JX,JY)g(\operatorname{grad}\alpha^v, JZ) - \\
&- g(\operatorname{grad}\alpha^v, JY)g(JX,JZ).
\end{aligned}
$$

By the substitution $Y := S_0$ (S_0 is an arbitrary semispray on M), we obtain

$$
\begin{aligned}
\text{(3.3d)}\qquad g\left(\left(\overline{h}-h\right)X, JZ\right) &= \alpha^c g(JX,JZ) - g(\mathcal{C}(X,Z), \overline{S} - S) - \\
&- g(JX,C)g(\operatorname{grad}\alpha^v, JZ).
\end{aligned}
$$

If $X := S_0$, (3.3d) implies the relation

$$
g(\overline{S} - S, JZ) = \alpha^c g(C, JZ) - 2Eg(\operatorname{grad}\alpha^v, JZ).
$$

Hence the semispray $\overline{S}$ associated with $\overline{h}$ and the canonical spray S are related as follows:

$$
\text{(3.3.e)}\qquad \overline{S} = S + \alpha^c C - 2E\operatorname{grad}\alpha^v.
$$

Substituting this into (3.3d) and applying the total symmetry of $\mathcal{C}_b$, we get the relation

$$
\begin{aligned}
(\overline{h}-h)X &= \alpha^c JX + 2E\mathcal{C}(F\operatorname{grad}\alpha^v, X) - d_J E(X)\operatorname{grad}\alpha^v \overset{(1.2d)}{=} \\
&= \alpha^c JX - E[J, \operatorname{grad}\alpha^v]X - d_J E(X)\operatorname{grad}\alpha^v. \qquad \square
\end{aligned}
$$

3.4. Corollary. *The tension of a Wagner endomorphism vanishes.*

Proof. Applying the formulas I.(1.6f)–(1.6i), a routine calculation shows that

$$\overline{H} = \mathcal{L}_C d_J E \otimes \operatorname{grad} \alpha^v + [C, \operatorname{grad} \alpha^v] \otimes d_J E \overset{\text{I.(3.3d);(1.2b)}}{=} 0. \qquad \square$$

3.5. Corollary. *The weak torsion and the strong torsion of a Wagner endomorphism can be given as follows:*

$$\overline{t} = d\alpha^v \otimes J - J \otimes d\alpha^v, \qquad \overline{T} = \alpha^c J - d\alpha^v \otimes C.$$

Proof. Applying the formulas I.(1.6g) and I.(1.6i), we get:

$$\overline{t} = d_J \alpha^c \wedge J - E[J, [J, \operatorname{grad} \alpha^v]].$$

From the graded Jacobi identity

$$[J, [J, \operatorname{grad} \alpha^v]] = [J, [\operatorname{grad} \alpha^v, J]] - [\operatorname{grad} \alpha^v, [J, J]] \overset{\text{I.(1.8c)}}{=}$$
$$= [J, [\operatorname{grad} \alpha^v, J]] = -[J, [J, \operatorname{grad} \alpha^v]],$$

therefore

$$[J, [J, \operatorname{grad} \alpha^v]] = 0.$$

Thus we have

$$\overline{t} = d_J \alpha^c \wedge J = d\alpha^v \otimes J - J \otimes d\alpha^v.$$

Finally, $\forall X \in \mathfrak{X}(M)$:

$$\overline{T}(X^c) := (i_{S_0} \overline{t} + \overline{H}) X^c \overset{\text{Cor. 3.4}}{=} \overline{t}(S_0, X^c) = (\alpha^c J - d\alpha^v \otimes C) X^c. \qquad \square$$

3.6. Corollary. *Let $\overline{h}$ be a Wagner endomorphism on M. Then*

$$d_{\overline{h}} \omega = \omega \wedge d\alpha^v.$$

Proof. We start from (3.3a). Since $d_h \omega = 0$ (see e.g. [10], p. 329), we have only to check the relation

$$d_{\alpha^c J - E[J, \operatorname{grad} \alpha^v] - d_J E \otimes \operatorname{grad} \alpha^v} \omega = \omega \wedge d\alpha^v.$$

Here

$$d_{\alpha^c J} \omega = (i_{\alpha^c J} \circ d - d \circ i_{\alpha^c J}) \omega = -d(\alpha^c i_J \omega) \overset{\text{I.(3.3c)}}{=} 0.$$

On the other hand, $\forall X, Y \in \mathfrak{X}(TM)$:

$$-\frac{1}{2} \left(i_{[J, \operatorname{grad} \alpha^v]} \omega \right) (X, Y) = -\frac{1}{2} \Big(\omega([J, \operatorname{grad} \alpha^v] X, Y) +$$
$$+ \omega(X, [J, \operatorname{grad} \alpha^v] Y) \Big) \overset{(1.2d)}{=} \omega(\mathcal{C}(F \operatorname{grad} \alpha^v, X), Y) -$$
$$- \omega(\mathcal{C}(F \operatorname{grad} \alpha^v, Y), X) = g(\mathcal{C}(F \operatorname{grad} \alpha^v, X), JY) -$$
$$- g(\mathcal{C}(F \operatorname{grad} \alpha^v, Y), JX) \overset{\text{I.3.7}}{=} 0,$$

therefore

$$d_{E[J,\operatorname{grad}\alpha^v]}\omega = 0.$$

Finally, $\forall X, Y \in \mathfrak{X}(TM)$:

$$\begin{aligned}(i_{d_J E\otimes \operatorname{grad}\alpha^v}\omega)(X,Y) &= \omega(JX(E)\operatorname{grad}\alpha^v, Y)+\\ &+\omega(X, JY(E)\operatorname{grad}\alpha^v) = (d_J E\otimes d\alpha^v)(X,Y)-\\ &-(d_J E\otimes d\alpha^v)(Y,X) = (d_J E\wedge d\alpha^v)(X,Y),\end{aligned}$$

so we get

$$d_{\overline{h}}\omega = -d_{d_J E\otimes \operatorname{grad}\alpha^v}\omega = d\left(i_{d_J E\otimes \operatorname{grad}\alpha^v}\omega\right) = d(d_J E\wedge d\alpha^v) = \omega\wedge d\alpha^v. \qquad \square$$

3.7. Proposition. *The second Cartan tensor $\overline{\mathcal{C}}'$ of a Wagner endomorphism $\overline{h}$ has the following properties:*

(3.7a) *it is semibasic,*

(3.7b) *its lowered tensor $\overline{\mathcal{C}}'_b$ is totally symmetric,*

(3.7c) $\overline{\mathcal{C}}'^{\circ} := i_{S_0}\overline{\mathcal{C}}' = 0$ *(S_0 is an arbitrary semispray on M).*

Proof. From the formula I.(3.4c) we get immediately the property (3.7a) and it is also clear that $\overline{\mathcal{C}}'_b$ is symmetric in its 2nd and 3rd arguments.

Evaluating the form $d_{\overline{h}}\,\omega$ on the vector fields $X^{\overline{h}}$, Y^v, $Z^{\overline{h}}$ ($X, Y, Z \in \mathfrak{X}(M)$) it follows that

$$\begin{aligned} d_{\overline{h}}\omega\left(X^{\overline{h}}, Y^v, Z^{\overline{h}}\right) &= 2\overline{\mathcal{C}}'_b\left(Z^{\overline{h}}, Y^{\overline{h}}, X^{\overline{h}}\right) - 2\overline{\mathcal{C}}'_b\left(X^{\overline{h}}, Y^{\overline{h}}, Z^{\overline{h}}\right) +\\ &+ (\omega\wedge d\alpha^v)\left(X^{\overline{h}}, Y^v, Z^{\overline{h}}\right) \overset{\text{Cor. 3.6}}{\Longrightarrow}\\ &\overline{\mathcal{C}}'_b\left(X^{\overline{h}}, Y^{\overline{h}}, Z^{\overline{h}}\right) = \overline{\mathcal{C}}'_b\left(Z^{\overline{h}}, Y^{\overline{h}}, X^{\overline{h}}\right),\end{aligned}$$

i.e. $\overline{\mathcal{C}}'_b$ is symmetric in its 1st and 3rd arguments.

According to the above symmetry properties, $\forall X, Y, Z \in \mathfrak{X}(M)$:

$$\overline{\mathcal{C}}'_b\left(X,^{\overline{h}}, Y^{\overline{h}}, Z^{\overline{h}}\right) = \overline{\mathcal{C}}'_b\left(Z^{\overline{h}}, Y^{\overline{h}}, X^{\overline{h}}\right) = \overline{\mathcal{C}}'_b\left(Z^{\overline{h}}, X^{\overline{h}}, Y^{\overline{h}}\right) = \overline{\mathcal{C}}'_b\left(Y^{\overline{h}}, X^{\overline{h}}, Z^{\overline{h}}\right),$$

i.e. $\overline{\mathcal{C}}'_b$ is totally symmetric.

Finally, let S_0 be an arbitrary semispray on M. Then $\forall Y, Z \in \mathfrak{X}(M)$:

$$\begin{aligned} 2g(\overline{\mathcal{C}}'(S_0, Y^{\overline{h}}), Z^v) &\overset{(3.7b)}{=} 2g(\overline{\mathcal{C}}'(Y^{\overline{h}}, S_0), Z^v) := Y^{\overline{h}}g(C, Z^v)-\\ &- g([Y^{\overline{h}}, C], Z^v) - g(C, [Y^{\overline{h}}, Z^v]) \overset{\text{I.(3.2a),I.(3.3c);Cor. 3.4}}{=}\\ &= Y^{\overline{h}}(Z^v(E)) - [Y^{\overline{h}}, Z^v](E) \overset{\text{Prop. 3.2}}{=} 0. \end{aligned} \qquad \square$$

3.8. Proposition. *Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E). The covariant derivatives with respect to $\overline{D}$ can explicitly be calculated by the following formulas:*

$$\overline{D}_{JX} JY = J[JX, Y] + \mathcal{C}(X, Y), \tag{3.8a}$$

$$\overline{D}_{\overline{h}X} JY = \overline{v}[\overline{h}X, JY] + \overline{\mathcal{C}}'(X, Y), \tag{3.8b}$$

$$\overline{D}_{JX} \overline{h}Y = \overline{h}[JX, Y] + \overline{F}\mathcal{C}(X, Y), \tag{3.8c}$$

$$\overline{D}_{\overline{h}X} \overline{h}Y = \overline{h}\,\overline{F}[\overline{h}X, JY] + \overline{F}\,\overline{\mathcal{C}}'(X, Y). \tag{3.8d}$$

Proof. Applying the usual "Christoffel process" it can easily be seen that a Wagner connection is uniquely determined by the conditions (3.1b)–(3.1e).

Consider now the Wagner endomorphism $\overline{h}$ and let us *define* a Finsler connection $(\overline{D}, \overline{h})$ by the formulas (3.8a)–(3.8d). It is easy to check that $(\overline{D}, \overline{h})$ satisfies the conditions (3.1a)–(3.1e) and, consequently, (3.8a)–(3.8d) are just the rules of calculation with respect to the Wagner connection $(\overline{D}, \overline{h}, \alpha)$. □

3.9. Remark. Comparing the formulas (3.8a)–(3.8d) with I.(4.5d)–(4.5g) we can say that a Wagner connection is a *"Cartan connection with nonvanishing $(h)h$-torsion"*, i.e. it is a *generalized Cartan connection.*

Our next Proposition emphasizes the strict analogy between the Cartan connection and a Wagner connection.

3.10. Lemma. $\forall X, Y \in \mathfrak{X}(TM)$*:*

$$\begin{aligned} \overline{D}_{\overline{h}X} JY - D_{\overline{h}X} JY = {} & g(JX, JY)\operatorname{grad}\alpha^v - g(\operatorname{grad}\alpha^v, JY)JX + \\ & + \mathcal{C}_b(F\operatorname{grad}\alpha^v, X, Y)C - JY(E)\mathcal{C}(F\operatorname{grad}\alpha^v, X) + \\ & + 2E\mathbb{Q}(F\operatorname{grad}\alpha^v, X)Y, \end{aligned} \tag{3.10a}$$

$$\begin{aligned} \overline{\mathcal{C}}'(X, Y) = {} & \mathcal{C}'(X, Y) + \alpha^c\mathcal{C}(X, Y) + JX(E)\mathcal{C}(F\operatorname{grad}\alpha^v, X) + \\ & + JY(E)\mathcal{C}(F\operatorname{grad}\alpha^v, X) + \mathcal{C}_b(F\operatorname{grad}\alpha^v, X, Y)C + \\ & + 2E(D_{\operatorname{grad}\alpha^v}\mathcal{C})(X, Y). \end{aligned} \tag{3.10b}$$

Since these relations can be obtained by an easy calculation from (3.3a) and (3.3c) we omit the proof. (Note that (3.10b) also implies Proposition 3.7)

3.11. Proposition. *Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection. Then the covariant differentials $\overline{D}\mathcal{C}$, $\overline{D}\,\overline{\mathcal{C}}'$ have the following properties:*

$$\overline{D}_{\overline{S}}\mathcal{C} = -\overline{\mathcal{C}}', \qquad \overline{D}_C\mathcal{C} = -\mathcal{C}, \qquad \overline{D}_C\overline{\mathcal{C}}' = 0. \tag{3.11}$$

Proof. According to the formulas I.(4.8a) and I.(4.8b), (3.11) immediately follows from the relations (3.10a), (3.10b). □

3.12. Proposition. *Let* $(\overline{D}, \overline{h}, \alpha)$ *be a Wagner connection on the Finsler manifold* (M, E). *Then the following assertions are equivalent:*

(a) $d\alpha^v = 0$ *(i.e.* $\alpha \in C^\infty(M)$ *is constant).*

(b) $\overline{S} = S$ *(i.e. the semispray* $\overline{S}$ *associated with* $\overline{h}$ *coincides with the canonical spray).*

(c) *The Wagner endomorphism arises from a semispray, i.e. there is a semispray* $\overline{S}$ *on* M *such that*

$$\overline{h} = \frac{1}{2}(1 + [J, \overline{S}]).$$

(d) $\overline{h} = h$ *(i.e. the Wagner endomorphism coincides with the Barthel endomorphism).*

(e) *The Finsler connection* $(\overline{D}, \overline{h})$ *coincides with the Cartan connection* (D, h).

Proof. (a) $\Longrightarrow$ *(e)* If $d\alpha^v = 0$ then it follows, by Corollary 3.5, that the weak torsion of $\overline{h}$ vanishes. Therefore the Wagner endomorphism is a conservative horizontal endomorphism on M with vanishing strong torsion (cf. Corollary 3.5). Thus $\overline{h} = h$ and (e) is an immediate consequence of (3.8a)–(3.8d).

The implications *(e)* $\Longrightarrow$ *(d)* and *(d)* $\Longrightarrow$ *(c)* are evident.

(c) $\Longrightarrow$ *(b)* It is easy to check that the hypothesis (c) implies the vanishing of the weak torsion $\overline{t}$. Hence, as above, $\overline{h} = h$ and consequently $\overline{S} = S$.

(b) $\Longrightarrow$ *(a)* If (b) holds then (3.3e) implies the relation

$$\operatorname{grad} \alpha^v = \mu C \qquad \left(\mu = \frac{\alpha^c}{2E} \in C^\infty(\mathcal{T}M)\right).$$

In view of (1.2e) this means that $\operatorname{grad} \alpha^v = 0$, proving the implication *(b)* $\Longrightarrow$ *(a)*. $\square$

4. Basic curvature identities

4.1. Lemma. *Let* $(\overline{D}, \overline{h}, \alpha)$ *be a Wagner connection on the Finsler manifold* (M, E). *There is a unique Finsler connection* $(\overset{\circ}{\overline{D}}, \overline{h})$ *on* M *such that:*

(4.1a) *the* $(v)hv$*-torsion* $\overset{\circ}{\overline{\mathbb{P}}}{}^1$ *of* $\overset{\circ}{\overline{D}}$ *vanishes:* $\overset{\circ}{\overline{\mathbb{P}}}{}^1 = 0$,

(4.1b) *the* $(h)hv$*-torsion* $\overset{\circ}{\overline{\mathbb{B}}}$ *of* $\overset{\circ}{\overline{D}}$ *vanishes:* $\overset{\circ}{\overline{\mathbb{B}}} = 0$.

The covariant derivatives with respect to $\overset{\circ}{\overline{D}}$ *can explicitly be calculated by the formulas*

$$\overset{\circ}{\overline{D}}_{JX} JY = J[JX, Y], \tag{4.1c}$$

$$\overset{\circ}{\overline{D}}_{\overline{h}X} JY = \overline{v}[\overline{h}X, JY], \tag{4.1d}$$

$$\overset{\circ}{\overline{D}}_{JX} \overline{h}Y = \overline{h}[JX, Y], \tag{4.1e}$$

$$\overset{\circ}{\overline{D}}_{\overline{h}X} \overline{h}Y = \overline{h}\,\overline{F}[\overline{h}X, JY]. \tag{4.1f}$$

In addition, $(\overset{\circ}{\overline{D}}, \overline{h})$ *has the following two properties:*

(4.1g) *the* h*-deflection* $\overline{h}^*(\overset{\circ}{\overline{D}}C)$ *of* $\overset{\circ}{\overline{D}}$ *vanishes:* $\overline{h}^*(\overset{\circ}{\overline{D}}C) = 0$,

(4.1h) $\overset{\circ}{\overline{D}}$ *is* $(h)h$*-semisymmetric, i.e. the* $(h)h$*-torsion* $\overset{\circ}{\overline{\mathbb{A}}}$ *of* $\overset{\circ}{\overline{D}}$ *has the following form:*

$$\overset{\circ}{\overline{\mathbb{A}}} = d\alpha^v \otimes \overline{h} - \overline{h} \otimes d\alpha^v.$$

Proof. We can argue as in the proof of Theorem 1 in [33]. □

4.2. Remark. It is easy to check (see e.g. [11], [33]) that if $\overline{h}$ coincides with the Barthel endomorphism then $(\overset{\circ}{\overline{D}}, \overline{h})$ is the well-known Berwald connection on the Finsler manifold (M, E). In general we can say that $(\overset{\circ}{\overline{D}}, \overline{h})$ is a *"Berwald connection with nonvanishing* $(h)h$*-torsion"*, i.e. it is a *"generalized Berwald connection"*.

4.3.Proposition. *Under the conditions of Lemma 4.1 the curvature tensors of* $\overline{D}$ *and* $\overset{\circ}{\overline{D}}$ *are related as follows:*

$$\begin{aligned}\overline{\mathbb{R}}(X,Y)Z = {} & \overset{\circ}{\overline{\mathbb{R}}}(X,Y)Z + \left(\overline{D}_{\overline{h}X}\overline{\mathcal{C}}'\right)(Y,Z) - \left(\overline{D}_{\overline{h}Y}\overline{\mathcal{C}}'\right)(X,Z) + \\ & + \overline{\mathcal{C}}'(\overline{F}\,\overline{\mathcal{C}}'(X,Z), Y) - \overline{\mathcal{C}}'(X, \overline{F}\,\overline{\mathcal{C}}'(Y,Z)) + \\ & + \overline{\mathcal{C}}'\left(\overline{F}\,\overline{t}(X,Y), Z\right) + \mathcal{C}\left(\overline{F}\,\overline{\Omega}(X,Y), Z\right),\end{aligned} \tag{4.3a}$$

$$\overline{\mathbb{P}}(X,Y)Z = \overset{\circ}{\overline{\mathbb{P}}}(X,Y)Z + \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z) - \left(\overline{D}_{JY}\overline{\mathcal{C}}'\right)(X,Z) + \tag{4.3b}$$

$$+\,\mathcal{C}(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z)-\overline{\mathcal{C}}'\left(X,\overline{F}\mathcal{C}(Y,Z)\right)+$$
$$+\,\mathcal{C}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z))-\overline{\mathcal{C}}'\left(\overline{F}\mathcal{C}(X,Y),Z\right),$$

(4.3c) $$\overline{\mathbb{Q}}(X,Y)Z=\mathcal{C}\left(\overline{F}\mathcal{C}(X,Z),Y\right)-\mathcal{C}\left(X,\overline{F}\mathcal{C}(Y,Z)\right),$$
$$\overset{\circ}{\overline{\mathbb{Q}}}=0\qquad(X,Y,Z\in\mathfrak{X}(TM)).$$

The *proof* is a straigthforward but lengthy calculation.

(Note that $\overline{\mathcal{C}}'\,(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z),\overline{\mathcal{C}}'(\overline{F}\,\overline{t}(X,Y),Z)\dots$ are independent of the choice of the almost complex structure $\overline{F}$.)

4.4. Corollary. *Let $(\overline{D},\overline{h},\alpha)$ be a Wagner connection. Then its curvature tensors have the following properties:*

(4.4a) $$\overline{\mathbb{R}}(X,Y)S_0=\overline{\Omega}(X,Y),$$

(4.4b) $$\overline{\mathbb{P}}(X,Y)S_0=\overline{\mathcal{C}}'(X,Y),\ \overline{\mathbb{P}}(X,S_0)Y=\overline{\mathbb{P}}(S_0,X)Y=0,$$

(4.4c) $$\overline{\mathbb{Q}}(X,Y)S_0=\overline{\mathbb{Q}}(X,S_0)Y=\overline{\mathbb{Q}}(S_0,X)Y=0$$
$(X,Y\in(TM),$ *S_0 is an arbitrary semispray on M*$)$.

Proof. We deduce only the less trivial third relation of (4.4b). Let $X,Y\in\mathfrak{X}(M)$ be arbitrary vector fields on M. Then

$$\overline{\mathbb{P}}\left(S_0,X^{\overline{h}}\right)Y^{\overline{h}}\overset{(4.3b)}{=}\overset{\circ}{\overline{\mathbb{P}}}\left(S_0,X^{\overline{h}}\right)Y^{\overline{h}}\overset{(4.1c),\,(4.1d)}{=}$$
$$=-\overset{\circ}{\overline{D}}_{X^v}\left(\overline{v}[\overline{S},Y^v]\right)-\overset{\circ}{\overline{D}}_{\overline{v}[\overline{S},X^v]}Y^v-\overset{\circ}{\overline{D}}_{\overline{h}[\overline{S},X^v]}Y^v=$$
$$\overset{\mathrm{I}.(2.6c),\mathrm{I}.(2.7a);(4.1c)}{=}\overset{\circ}{\overline{D}}_{X^v}\left(Y^{\overline{h}}-Y^c-\overline{T}(Y^c)\right)-\overset{\circ}{\overline{D}}_{\overline{F}J[\overline{S},X^v]}Y^v=$$
$$=\overset{\circ}{\overline{D}}_{X^v}\left(Y^{\overline{h}}-Y^c\right)-\overset{\circ}{\overline{D}}_{X^v}(\overline{T}(Y^c))-\overset{\circ}{\overline{D}}_{\overline{F}J[\overline{S},X^v]}Y^v.$$

Here

$$\overset{\circ}{\overline{D}}_{X^v}\left(Y^{\overline{h}}-Y^c\right)=[X^v,Y^{\overline{h}}-Y^c],\quad\text{since the }(v)v\text{-torsion of }\overset{\circ}{\overline{D}}\text{ vanishes,}$$

$$\overset{\circ}{\overline{D}}_{X^v}(\overline{T}(Y^c))\overset{\mathrm{Cor.}\ 3.5,\ (4.1c)}{=}(X\alpha)^vY^v-(Y\alpha)^v\overset{\circ}{\overline{D}}_{X^v}C\overset{(4.1c)}{=}$$
$$=(X\alpha)^vY^v-(Y\alpha)^vJ[X^v,\overline{S}]\overset{\mathrm{I}.1.15}{=}$$
$$=(X\alpha)^vY^v-(Y\alpha)^vX^v\overset{\mathrm{Cor.}\ 3.5}{=}\overline{t}\left(X^{\overline{h}},Y^{\overline{h}}\right),$$

$$\overset{\circ}{\overline{D}}_{\overline{F}J[\overline{S},X^v]}Y^v\overset{\mathrm{I}.1.15}{=}\overset{\circ}{\overline{D}}_{-\overline{F}X^v}Y^v=\overset{\circ}{\overline{D}}_{-X^{\overline{h}}}Y^v\overset{(4.1d)}{=}-[X^{\overline{h}},Y^v].$$

Thus we have:

$$\mathbb{P}\left(S_0,X^{\overline{h}}\right)Y^{\overline{h}}=[X^v,Y^{\overline{h}}-Y^c]-\overline{t}\left(X^{\overline{h}},Y^{\overline{h}}\right)+[X^{\overline{h}},Y^v]\overset{\mathrm{I}.(2.6b),\mathrm{I}(1.16a)}{=}0.\qquad\square$$

4.5. Lemma. – $\forall X,Y,Z \in \mathfrak{X}(TM)$:

$$\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{\mathbb{A}}\left(\overline{\mathbb{A}}(X,Y),Z\right)-\left(\overline{D}_{\overline{h}X}\overline{\mathbb{A}}\right)(Y,Z)\right)=0.$$

Proof. Omitting the troublesome details we note that $\forall X,Y,Z \in \mathfrak{X}(M)$:

$$\begin{aligned}
&\underset{X^{\overline{h}},Y^{\overline{h}},Z^{\overline{h}}}{\mathfrak{S}}\left(\overline{\mathbb{A}}\left(\overline{\mathbb{A}}\left(X^{\overline{h}},Y^{\overline{h}}\right),Z^{\overline{h}}\right)+\left(\overline{D}_{X^{\overline{h}}}\overline{\mathbb{A}}\right)\left(Y^{\overline{h}},Z^{\overline{h}}\right)\right):=\\
&=\overline{\mathbb{A}}\left(\overline{\mathbb{A}}\left(X^{\overline{h}},Y^{\overline{h}}\right),Z^{\overline{h}}\right)+\overline{\mathbb{A}}\left(\overline{\mathbb{A}}\left(Y^{\overline{h}},Z^{\overline{h}}\right),X^{\overline{h}}\right)+\overline{\mathbb{A}}\left(\overline{\mathbb{A}}\left(Z^{\overline{h}},X^{\overline{h}}\right),Y^{\overline{h}}\right)+\\
&+\left(\overline{D}_{X^{\overline{h}}}\overline{\mathbb{A}}\right)\left(Y^{\overline{h}},Z^{\overline{h}}\right)+\left(\overline{D}_{Y^{\overline{h}}}\overline{\mathbb{A}}\right)\left(Z^{\overline{h}},X^{\overline{h}}\right)+\left(\overline{D}_{Z^{\overline{h}}}\overline{\mathbb{A}}\right)\left(X^{\overline{h}},Y^{\overline{h}}\right)=\\
&=(Y\alpha)^v\overline{\mathbb{A}}\left(X^{\overline{h}},Z^{\overline{h}}\right)-(X\alpha)^v\overline{\mathbb{A}}\left(Y^{\overline{h}},Z^{\overline{h}}\right)-(Z\alpha)^v\overline{\mathbb{A}}\left(X^{\overline{h}},Y^{\overline{h}}\right)=\\
&=-(d\alpha^v\wedge\overline{\mathbb{A}})\left(X^{\overline{h}},Y^{\overline{h}},Z^{\overline{h}}\right)\overset{(3.1d)}{=}0.
\end{aligned}$$

□

4.6. Corollary. *(Bianchi identities)* –$\forall X,Y,Z \in \mathfrak{X}(TM)$:

(4.6a) $$\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{D}_{\overline{h}X}\overline{\Omega}\right)(Y,Z)=\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{\mathcal{C}}'\left(\overline{F}\,\overline{\Omega}(X,Y),Z\right)-\overline{\Omega}\left(\overline{\mathbb{A}}(X,Y),Z\right)\right),$$

(4.6b) $$\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{D}_{\overline{h}X}\overline{\mathbb{R}}\right)(Y,Z)=\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{\mathbb{P}}\left(X,\overline{F}\,\overline{\Omega}(Y,Z)\right)-\overline{\mathbb{R}}\left(\overline{\mathbb{A}}(X,Y),Z\right)\right),$$

(4.6c) $$\underset{X,Y,Z}{\mathfrak{S}}\left(\overline{D}_{JX}\overline{\mathbb{Q}}\right)(Y,Z)=0,$$

(4.6d) $$\begin{aligned}
&\left(\overline{D}_{\overline{h}X}\overline{\mathbb{P}}\right)(Y,Z)-\left(\overline{D}_{\overline{h}Y}\overline{\mathbb{P}}\right)(X,Z)+\left(\overline{D}_{JZ}\overline{\mathbb{R}}\right)(X,Y)=\\
&=\overline{\mathbb{P}}(X,\overline{F}\,\overline{\mathcal{C}}'(Y,Z))-\overline{\mathbb{P}}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z))-\overline{\mathbb{R}}(X,\overline{F}\mathcal{C}(Y,Z))+\\
&+\overline{\mathbb{R}}(Y,\overline{F}\mathcal{C}(X,Z))-\overline{\mathbb{P}}\left(\overline{\mathbb{A}}(X,Y),Z\right)-\overline{\mathbb{Q}}\left(\overline{F}\,\overline{\Omega}(X,Y),Z\right),
\end{aligned}$$

(4.6e) $$\begin{aligned}
&\left(\overline{D}_{\overline{h}X}\overline{\mathbb{Q}}\right)(Y,Z)-\left(\overline{D}_{JY}\overline{\mathbb{P}}\right)(X,Z)+\left(\overline{D}_{JZ}\overline{\mathbb{P}}\right)(X,Y)=\\
&=\overline{\mathbb{P}}\left(\overline{F}\mathcal{C}(X,Y),Z\right)-\overline{\mathbb{P}}(\overline{F}\mathcal{C}(Z,X),Y)-\overline{\mathbb{Q}}(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z)+\\
&+\overline{\mathbb{Q}}(\overline{F}\,\overline{\mathcal{C}}'(Z,X),Y).
\end{aligned}$$

Proof. Let $X,Y,Z \in \mathfrak{X}(TM)$ be arbitrary. By Lemma 4.5, the "usual" first Bianchi identity

$$\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\overline{\mathbb{K}}\left(\overline{h}X,\overline{h}Y\right)\overline{h}Z=\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\left(\overline{\mathbb{T}}\left(\overline{\mathbb{T}}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)+\left(\overline{D}_{\overline{h}X}\overline{\mathbb{T}}\right)\left(\overline{h}Y,\overline{h}Z\right)\right)$$

gives the relation

$$\begin{aligned}
\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\overline{\mathbb{K}}\left(\overline{h}X,\overline{h}Y\right)\overline{h}Z=\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\Big(&\left(\overline{D}_{\overline{h}X}\overline{\Omega}\right)\left(\overline{h}Y,\overline{h}Z\right)+\\
&+\overline{\Omega}\left(\overline{\mathbb{A}}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)+\overline{F}\mathcal{C}\left(\overline{F}\,\overline{\Omega}\left(\overline{h}X,\overline{h}Y\right)\overline{h}Z\right)-\\
&-\overline{\mathcal{C}}'\left(\overline{F}\,\overline{\Omega}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)\Big),
\end{aligned}$$

since

$$\overline{\mathbb{T}}\left(\overline{h}X,\overline{h}Y\right)=\overline{\mathbb{A}}(X,Y)+\overline{\Omega}(X,Y), \tag{4.6f}$$

$$\overline{\mathbb{T}}\left(\overline{h}X,JY\right)=\overline{\mathcal{C}}'(X,Y)-\overline{F}\mathcal{C}(X,Y). \tag{4.6g}$$

From this it follows that

$$\begin{aligned}0=\overline{v}&\left(\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\overline{\mathbb{K}}\left(\overline{h}X,\overline{h}Y\right)\overline{h}Z\right)=\\&=\underset{\overline{h}X,\overline{h}Y,\overline{h}Z}{\mathfrak{S}}\Big(\left(\overline{D}_{\overline{h}X}\overline{\Omega}\right)\left(\overline{h}Y,\overline{h}Z\right)+\overline{\Omega}\left(\overline{\mathbb{A}}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)-\\&\quad-\overline{\mathcal{C}}'\left(\overline{F}\,\overline{\Omega}\left(\overline{h}X,\overline{h}Y\right),\overline{h}Z\right)\Big),\end{aligned}$$

which proves the relation (4.6a).

Applying the second Bianchi identity, the other relations can also be obtained by a direct calculation. For example we derive (4.6d). Since

$$0=\underset{\overline{h}X,\overline{h}Y,JZ}{\mathfrak{S}}\Big(\overline{\mathbb{K}}\left(\overline{\mathbb{T}}\left(\overline{h}X,\overline{h}Y\right),JZ\right)+\left(\overline{D}_{\overline{h}X}\overline{\mathbb{K}}\right)\left(\overline{h}Y,JZ\right)\Big),$$

we get

$$\begin{aligned}0=&\,\overline{\mathbb{P}}\left(\overline{\mathbb{A}}(X,Y),Z\right)+\overline{\mathbb{Q}}\left(\overline{F}\,\overline{\Omega}(X,Y),Z\right)-\overline{\mathbb{P}}(X,\overline{F}\,\overline{\mathcal{C}}'(Y,Z))-\\&-\overline{\mathbb{R}}\left(\overline{F}\mathcal{C}(Y,Z),X\right)+\overline{\mathbb{P}}(Y,F\overline{\mathcal{C}}'(X,Z))+\overline{\mathbb{R}}\left(\overline{F}\mathcal{C}(X,Z),Y\right)+\\&+\left(\overline{D}_{\overline{h}X}\overline{\mathbb{P}}\right)(Y,Z)-\left(\overline{D}_{\overline{h}Y}\overline{\mathbb{P}}\right)(X,Z)+\left(\overline{D}_{JZ}\overline{\mathbb{R}}\right)(X,Y),\end{aligned}$$

proving the relation (4.6d). □

4.7. Corollary. $\forall X,Y,Z\in\mathfrak{X}(TM)$*:*

$$\left(\overline{D}_C\overline{\mathbb{R}}\right)(X,Y)Z=0,\qquad\left(\overline{D}_C\overline{\mathbb{P}}\right)(X,Y)Z=-\overline{\mathbb{P}}(X,Y)Z, \tag{4.7a}$$
$$\left(\overline{D}_C\overline{\mathbb{Q}}\right)(X,Y)Z=-2\overline{\mathbb{Q}}(X,Y)Z,$$

$$\begin{aligned}\left(\overline{D}_{\overline{S}}\overline{\mathbb{Q}}\right)(X,Y)Z=&\,\mathcal{C}(X,\overline{F}\,\overline{\mathcal{C}}'(Y,Z))-\mathcal{C}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z))+\\&+\overline{\mathcal{C}}'\left(X,\overline{F}\mathcal{C}(Y,Z)\right)-\overline{\mathcal{C}}'\left(Y,\overline{F}\mathcal{C}(X,Z)\right).\end{aligned} \tag{4.7b}$$

Proof. Substituting $Z:=\overline{S}$ in the Bianchi identity (4.6d), we have

$$\overline{D}_C\overline{\mathbb{R}}=0.$$

In the same way, consider the vector field $Y:=\overline{S}$. From the Bianchi identity (4.6e) it follows that

$$\overline{D}_C\overline{\mathbb{P}}=-\overline{\mathbb{P}}.$$

The relation $\overline{D}_C\overline{\mathbb{Q}}=-2\overline{\mathbb{Q}}$ is an immediate consequence of the Bianchi identity (4.6c).

Finally, by Proposition 3.11 and (4.3c), an easy direct calculation shows that (4.7b) holds. □

4.8. Corollary. *$\forall X, Y, Z, W \in \mathfrak{X}(TM)$:*

$$g\left(\overline{\mathbb{P}}(X,Y)Z, JW\right) = -g\left(\overline{\mathbb{P}}(X,Y)W, JZ\right), \tag{4.8a}$$

$$\begin{aligned}\overline{\mathbb{P}}(X,Y)Z - \overline{\mathbb{P}}(Z,Y)X = {} & \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z) - \left(\overline{D}_{\overline{h}Z}\mathcal{C}\right)(X,Y) + \\ & + \mathcal{C}(\overline{F}\,\overline{\mathcal{C}}'(X,Y),Z) - \mathcal{C}(X,\overline{F}\,\overline{\mathcal{C}}'(Z,Y)),\end{aligned} \tag{4.8b}$$

$$\begin{aligned}\overline{\mathbb{P}}(X,Y)Z - \overline{\mathbb{P}}(X,Z)Y = {} & \left(\overline{D}_{JZ}\overline{\mathcal{C}}'\right)(X,Y) - \left(\overline{D}_{JY}\overline{\mathcal{C}}'\right)(X,Z) + \\ & + \overline{\mathcal{C}}'\left(\overline{F}\mathcal{C}(Z,X),Y\right) - \overline{\mathcal{C}}'\left(\overline{F}\mathcal{C}(X,Y),Z\right),\end{aligned} \tag{4.8c}$$

$$\begin{aligned}\overline{\mathbb{P}}(X,Y)Z - \overline{\mathbb{P}}(Y,X)Z = {} & \mathcal{C}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z)) - \mathcal{C}(X,\overline{F}\,\overline{\mathcal{C}}'(Y,Z)) + \\ & + \overline{\mathcal{C}}'\left(Y,\overline{F}\mathcal{C}(X,Z)\right) - \overline{\mathcal{C}}'\left(X,\overline{F}\mathcal{C}(Y,Z)\right).\end{aligned} \tag{4.8d}$$

Proof. It is easy to check that the first identity holds for the curvature tensors of an arbitrary metrical connection.

Let now $X, Y, Z \in \mathfrak{X}(M)$ be arbitrary vector fields on M. Applying the formulas (3.8a)–(3.8d) we get:

$$\begin{aligned}\overline{\mathbb{P}}\left(X^{\overline{h}}, Y^{\overline{h}}\right)Z^{\overline{h}} - \overline{\mathbb{P}}\left(Z^{\overline{h}}, Y^{\overline{h}}\right)X^{\overline{h}} = {} & \left(\overline{D}_{X^{\overline{h}}}\mathcal{C}\right)\left(Y^{\overline{h}}, Z^{\overline{h}}\right) - \left(\overline{D}_{Z^{\overline{h}}}\mathcal{C}\right)\left(X^{\overline{h}}, Y^{\overline{h}}\right) + \\ & + \mathcal{C}\left(\overline{F}\,\overline{\mathcal{C}}'\left(X^{\overline{h}}, Y^{\overline{h}}\right), Z^{\overline{h}}\right) - \mathcal{C}\left(X^{\overline{h}}, \overline{F}\,\overline{\mathcal{C}}'\left(Y^{\overline{h}}, Z^{\overline{h}}\right)\right).\end{aligned}$$

Evaluating the Bianchi identity (4.6e) on $\overline{S}$, it follows by Corollary 4.4 that (4.8c) holds.

Finally, from the Bianchi identity (4.6e), $\forall X, Y, Z \in \mathfrak{X}(TM)$:

$$\left(\overline{D}_{\overline{S}}\overline{\mathbb{Q}}\right)(X,Y)Z - \left(\overline{D}_{JX}\overline{\mathbb{P}}\right)(\overline{S},Y)Z + \left(\overline{D}_{JY}\overline{\mathbb{P}}\right)(\overline{S},X)Z = 0,$$

so we have

$$\begin{aligned}\overline{\mathbb{P}}(X,Y)Z - \overline{\mathbb{P}}(Y,X)Z &= -\left(\overline{D}_{\overline{S}}\overline{\mathbb{Q}}\right)(X,Y)Z \overset{(4.7b)}{=} \\ &= \mathcal{C}(Y,\overline{F}\,\overline{\mathcal{C}}'(X,Z)) - \mathcal{C}(X,\overline{F}\,\overline{\mathcal{C}}'(Y,Z)) + \\ &+ \overline{\mathcal{C}}'\left(Y,\overline{F}\mathcal{C}(X,Z)\right) - \overline{\mathcal{C}}'\left(X,\overline{F}\mathcal{C}(Y,Z)\right).\end{aligned}$$

□

5. Wagner manifolds

5.1. Definition. Let (M, E) be a Finsler manifold endowed with a Wagner connection $(\overline{D}, \overline{h}, \alpha)$. (M, E) is said to be a *Wagner manifold* (with respect to $(\overline{D}, \overline{h}, \alpha)$) if there is a linear connection ∇ on M such that

$$\forall X, Y \in \mathfrak{X}(M) : \overline{D}_{X^{\overline{h}}} Y^v = (\nabla_X Y)^v . \tag{5.1}$$

Then ∇ is called the *linear connection of the Wagner manifold.*

5.2. Proposition. *If (M, E) is a Wagner manifold (with respect to $(\overline{D}, \overline{h}, \alpha)$) then the second Cartan tensor $\overline{\mathcal{C}}'$ of $\overline{h}$ vanishes.*

Proof. Since (M, E) is a Wagner manifold, it follows that $\overline{D}_{X^{\overline{h}}} Z^v$ $(X, Z \in \mathfrak{X}(M))$ is a vertically lifted vector field to the manifold $\mathcal{T}M$ and, consequently, $\forall X, Y,$ $Z \in \mathfrak{X}(M)$:

$$\begin{aligned} 0 = [\overline{D}_{X^{\overline{h}}} Z^v, Y^v] &\overset{(3.1c)}{=} \overline{D}_{\left(\overline{D}_{X^{\overline{h}}} Z^v\right)} Y^v - \overline{D}_{Y^v} \overline{D}_{X^{\overline{h}}} Z^v \overset{(3.8a)}{=} \\ &= \mathcal{C}\left(\overline{F}\, \overline{D}_{X^{\overline{h}}} Z^v, Y^{\overline{h}}\right) + \overline{\mathbb{P}}\left(X^{\overline{h}}, Y^{\overline{h}}\right) Z^{\overline{h}} - \overline{D}_{X^{\overline{h}}}\left(\mathcal{C}\left(Y^{\overline{h}}, Z^{\overline{h}}\right)\right) + \\ &\quad + \mathcal{C}\left(\overline{F}[X^{\overline{h}}, Y^v], Z^{\overline{h}}\right) \overset{(3.8d)}{=} \overline{\mathbb{P}}\left(X^{\overline{h}}, Y^{\overline{h}}\right) Z^{\overline{h}} - (\overline{D}_{X^{\overline{h}}} \mathcal{C})\left(Y^{\overline{h}}, Z^{\overline{h}}\right) - \\ &\quad - \mathcal{C}\left(\overline{F}\, \overline{\mathcal{C}}'\left(X^{\overline{h}}, Y^{\overline{h}}\right), Z^{\overline{h}}\right), \end{aligned}$$

therefore (5.1) is equivalent to the relation

$$\overline{\mathbb{P}}(X, Y)Z - \left(\overline{D}_{\overline{h}X} \mathcal{C}\right)(Y, Z) - \mathcal{C}(\overline{F}\, \overline{\mathcal{C}}'(X, Y), Z) = 0, \tag{5.2}$$

where $X, Y, Z \in \mathfrak{X}(TM)$.

By the substitution $Z := \overline{S}$ we obtain that

$$0 = \overline{\mathbb{P}}(X, Y)\overline{S} \overset{(4.4b)}{=} \overline{\mathcal{C}}'(X, Y). \qquad \square$$

5.3. Theorem. *Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E). Then the following assertions are equivalent:*

(a) (M, E) is a Wagner manifold (with respect to $(\overline{D}, \overline{h}, \alpha)$).

(b) The hv-curvature tensor $\overset{\circ}{\overline{\mathbb{P}}}$ of the Finsler connection $(\overset{\circ}{\overline{D}}, \overline{h})$ vanishes.

Proof. *(a) $\Longrightarrow$ (b)* In view of (5.2) and (4.3b), we have

$$\begin{aligned} \overset{\circ}{\overline{\mathbb{P}}}(X, Y)Z = \left(\overline{D}_{JY} \overline{\mathcal{C}}'\right)(X, Z) + \overline{\mathcal{C}}'(X, \overline{F}\mathcal{C}(Y, Z)) + \overline{\mathcal{C}}'(\overline{F}\mathcal{C}(X, Y), Z) - \\ - \mathcal{C}(Y, \overline{F}\, \overline{\mathcal{C}}'(X, Z)) \overset{\text{Prop. 5.2}}{=} 0. \end{aligned}$$

(b) $\Longrightarrow$ (a) Since $\forall X, Y, Z \in \mathfrak{X}(M)$: $\overset{\circ}{\overline{\mathbb{P}}}\left(X^{\overline{h}}, Y^{\overline{h}}\right) Z^{\overline{h}} = [[X^{\overline{h}}, Y^v], Z^v]$, the vanishing of $\overset{\circ}{\overline{\mathbb{P}}}$ implies that $[X^{\overline{h}}, Y^v]$ is a vertical lift.

On the other hand,

$$2g\left(\overline{\mathcal{C}}'\left(X^{\overline{h}}, Y^{\overline{h}}\right), Z^v\right) = X^{\overline{h}} g(Y^v, Z^v) - g\left([X^{\overline{h}}, Y^v], Z^v\right) -$$
$$- g\left(Y^v, [X^{\overline{h}}, Z^v]\right) \overset{\mathrm{I.(3.3b)}}{=} X^{\overline{h}}(Y^v(Z^v E)) - [X^{\overline{h}}, Y^v](Z^v E) -$$
$$- Y^v\left([X^{\overline{h}}, Z^v]E\right) \overset{\text{Prop. 3.2}}{=} 0,$$

so the second Cartan tensor $\overline{\mathcal{C}}'$ of $\overline{h}$ vanishes, and consequently $\forall X, Y \in \mathfrak{X}(M)$:

$$\overline{D}_{X^{\overline{h}}} Y^v = \overset{\circ}{\overline{D}}_{X^{\overline{h}}} Y^v \overset{(4.1\mathrm{d})}{=} [X^{\overline{h}}, Y^v].$$

Finally, if we define a linear connection ∇ on M by the formula

$$(\nabla_X Y)^v := [X^{\overline{h}}, Y^v], \tag{5.3a}$$

then ∇ clearly satisfies the condition (5.1). $\square$

5.4. Proposition. *Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection. Then the following assertions are equivalent:*

(a) the hv-curvature tensor $\overline{\mathbb{P}}$ of $\overline{D}$ vanishes: $\overline{\mathbb{P}} = 0$.

(b) The second Cartan tensor $\overline{\mathcal{C}}'$ of $\overline{h}$ vanishes: $\overline{\mathcal{C}}' = 0$.

(c) $\forall X, Y, Z \in \mathfrak{X}(TM) : \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z) = \left(\overline{D}_{\overline{h}Z}\mathcal{C}\right)(X, Y)$.

(d) $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : \overset{\circ}{\overline{\mathbb{P}}}(X, Y)Z = -\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z)$.

Proof. From (4.4b) we immediately get the implication *(a)* $\Longrightarrow$ *(b)*.

This implies by (4.8b) that *(a)* $\Longrightarrow$ *(c)* is also valid.

(a) $\Longrightarrow$ *(d)* In view of (4.3b), we have

$$\overset{\circ}{\overline{\mathbb{P}}}(X, Y)Z + \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z) = \left(\overline{D}_{JY}\overline{\mathcal{C}}'\right)(X, Z) - \mathcal{C}(\overline{F}\,\overline{\mathcal{C}}'(X, Y), Z) +$$
$$+ \overline{\mathcal{C}}'(X, \overline{F}\mathcal{C}(Y, Z)) - \mathcal{C}(Y, \overline{F}\,\overline{\mathcal{C}}'(X, Z)) + \overline{\mathcal{C}}'(\overline{F}\mathcal{C}(X, Y), Z) \overset{(a) \Rightarrow (b)}{=} 0.$$

(d) $\Longrightarrow$ *(a)* As it can be seen from the proof of Corollary 4.4,

$$\forall X, Y \in \mathfrak{X}(M) : 0 \overset{(4.4\mathrm{b})}{=} \overline{\mathbb{P}}(\overline{S}, X^{\overline{h}})Y^{\overline{h}} = \overset{\circ}{\overline{\mathbb{P}}}(\overline{S}, X^{\overline{h}})Y^{\overline{h}},$$

where $\overline{S}$ is the semispray associated with $\overline{h}$. Therefore, by the substitution $X := \overline{S}$, (d) yields the relation

$$0 = \overset{\circ}{\overline{\mathbb{P}}}(\overline{S}, Y)Z = -\left(\overline{D}_{\overline{S}}\mathcal{C}\right)(Y, Z) \overset{\text{Prop. 3.11}}{=} \overline{\mathcal{C}}'(Y, Z).$$

Hence, by (4.3b),

$$\overline{\mathbb{P}}(X, Y)Z = 0.$$

(c) $\Longrightarrow$ *(a)* Let $X := \overline{S}$ in (c). Then

$$\left(\overline{D}_{\overline{S}}\mathcal{C}\right)(Y,Z) = \left(\overline{D}_{\overline{h}Z}\mathcal{C}\right)(\overline{S},Y) = 0, \text{ and consequently } \overline{\mathcal{C}}' = 0.$$

By Corollary 4.8, this means that the curvature tensor $\overline{\mathbb{P}}$ is totally symmetric. Since $\forall X,Y,Z \in \mathfrak{X}(TM)$:

$$\overline{\mathbb{P}}(X,Y)Z = \frac{1}{2}\Big(\overline{\mathbb{P}}(X+Z,Y)X+Z - \overline{\mathbb{P}}(X,Y)X - \overline{\mathbb{P}}(Z,Y)Z\Big)$$

and

$$-g(\overline{\mathbb{P}}(X,Y)X, JZ) \overset{(4.8a)}{=} g(\overline{\mathbb{P}}(X,Y)Z, JX) = g(\overline{\mathbb{P}}(Z,Y)X, JX) \overset{(4.8a)}{=} 0,$$

it follows that

$$\overline{\mathbb{P}} = 0.$$

(b) $\Longrightarrow$ *(a)* From the assumption $\overline{\mathcal{C}}' = 0$ and (4.8c), (4.8d), we get immediately that $\forall X,Y,Z \in \mathfrak{X}(TM)$:

$$\overline{\mathbb{P}}(X,Y)Z \overset{(4.8c)}{=} \overline{\mathbb{P}}(X,Z)Y \overset{(4.8d)}{=} \overline{\mathbb{P}}(Z,X)Y \overset{(4.8c)}{=} \overline{\mathbb{P}}(Z,Y)X,$$

i.e. $\overline{\mathbb{P}}$ is totally symmetric. So, repeating the preceding reasoning, we infer that $\overline{\mathbb{P}} = 0$. □

5.5. Theorem. *Let $(\overline{D},\overline{h},\alpha)$ be a Wagner connection on the Finsler manifold (M,E). Then the following assertions are equivalent:*

(a) (M,E) is a Wagner manifold (with respect to $(\overline{D},\overline{h},\alpha)$).

(b) $\forall X,Y,Z \in \mathfrak{X}(TM) : \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z) = 0$.

Proof. *(a)* $\Longrightarrow$ *(b)* We know from Proposition 5.2 that $\overline{\mathcal{C}}' = 0$ and so

$$\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z) \overset{\text{Prop. 5.4}}{=} -\overset{\circ}{\overline{\mathbb{P}}}(X,Y)Z \overset{\text{Th. 5.3}}{=} 0.$$

(b) $\Longrightarrow$ *(a)* Our assumption $\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z) = 0$ implies by Proposition 3.11 that

$$0 = -\left(\overline{D}_{\overline{S}}\mathcal{C}\right)(Y,Z) = \overline{\mathcal{C}}'(Y,Z).$$

Applying Proposition 5.4, this yields the relation

$$\overset{\circ}{\overline{\mathbb{P}}}(X,Y)Z = -\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y,Z),$$

therefore

$$\overset{\circ}{\overline{\mathbb{P}}}(X,Y)Z = 0,$$

so (M,E) is a Wagner manifold. □

5.6. Proposition. *Let (M, E) be a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$. Then the following assertions are equivalent:*

(a) $\overline{\Omega} = 0$ *(i.e., $\overline{h}$ is integrable).*

(b) *The h-curvature tensor $\overline{\mathbb{R}}$ of $\overline{D}$ vanishes.*

(c) *The h-curvature tensor $\overset{\circ}{\overline{\mathbb{R}}}$ of $\overset{\circ}{\overline{D}}$ vanishes.*

Proof. As it was shown in Proposition 5.2, the second Cartan tensor $\overline{\mathcal{C}}'$ belonging to $\overline{h}$ vanishes in any Wagner manifold. This means that the Bianchi identity (4.6d) reduces to the formula

$$\begin{aligned} &\left(\overline{D}_{\overline{h}X}\overline{\mathbb{P}}\right)(Y, Z) - \left(\overline{D}_{\overline{h}Y}\overline{\mathbb{P}}\right)(X, Z) + \left(\overline{D}_{JZ}\overline{\mathbb{R}}\right)(X, Y) \\ &= -\overline{\mathbb{R}}(X, \overline{F}\mathcal{C}(Y, Z)) + \overline{\mathbb{R}}(Y, \overline{F}\mathcal{C}(X, Z)) \\ &- \overline{\mathbb{P}}(\overline{\mathbb{A}}(X, Y), Z) - \overline{\mathbb{Q}}(\overline{F}\,\overline{\Omega}(X, Y), Z). \end{aligned} \tag{5.6a}$$

Substituting a semispray S_0 into (5.6a), by Corollary 4.4 we get the relation

$$\left(\overline{D}_{JZ}\overline{\Omega}\right)(X, Y) - \overline{\mathbb{R}}(X, Y)Z = -\overline{\Omega}(X, \overline{F}\mathcal{C}(Y, Z)) + \overline{\Omega}(Y, \overline{F}\mathcal{C}(X, Z)).$$

Therefore the implication (a) $\Longrightarrow$ (b) holds. The converse is an immediate consequence of the relation (4.4a).

Finally, by Proposition 4.3,

$$\overline{\mathbb{R}}(X, Y)Z = \overset{\circ}{\overline{\mathbb{R}}}(X, Y)Z + \mathcal{C}(\overline{F}\,\overline{\Omega}(X, Y), Z),$$

so (b) and (c) are also equivalent. (We can argue as in the proof of I.7.1 using the equivalence (a) $\Longleftrightarrow$ (b) and (4.4a).) $\square$

6. Hashiguchi-Ichijyō's theorems

6.1. Theorem. *Let (M,E) be a Wagner manifold with respect to $(\overline{D},\overline{h},\alpha)$ and let us consider the conformal change $\widetilde{g}=\varphi g$ $(\varphi=\exp\circ\beta^v)$ of the metric g. Then the Finsler manifold $(M,\widetilde{E})$ is also a Wagner manifold with respect to the Wagner connection induced by $\frac{1}{2}\beta+\alpha\in C^\infty(M)$.*

Proof. Let us consider the Wagner endomorphism $\overline{\overline{h}}$ induced by the function $\frac{1}{2}\beta+\alpha$.

We get from the relation (3.3a)

$$\overline{\overline{h}}=\widetilde{h}+\left(\frac{1}{2}\beta+\alpha\right)^c J-\widetilde{E}\big[J,\widetilde{\operatorname{grad}}\left(\frac{1}{2}\beta+\alpha\right)^v\big]-d_J\widetilde{E}\otimes\widetilde{\operatorname{grad}}\left(\frac{1}{2}\beta+\alpha\right)^v=$$

$$\overset{\text{Prop. 1.12}}{=}\widetilde{h}+\left(\frac{1}{2}\beta+\alpha\right)^c J-E\big[J,\operatorname{grad}\left(\frac{1}{2}\beta+\alpha\right)^v\big]-d_JE\otimes\operatorname{grad}\left(\frac{1}{2}\beta+\alpha\right)^v=$$

$$=\widetilde{h}+\frac{1}{2}\beta^cJ-\frac{1}{2}E[J,\operatorname{grad}\beta^v]-\frac{1}{2}d_JE\otimes\operatorname{grad}\beta^v+$$

$$+\alpha^cJ-E[J,\operatorname{grad}\alpha^v]-d_JE\otimes\operatorname{grad}\alpha^v\overset{\text{Cor. 2.2}}{=}h-\frac{1}{2}d\beta^v\otimes C+\alpha^cJ-$$

$$-E[J,\operatorname{grad}\alpha^v]-d_JE\otimes\operatorname{grad}\alpha^v\overset{(3.3a)}{=}\overline{h}-\frac{1}{2}d\beta^v\otimes C.$$

Using this form of $\overline{\overline{h}}$, we easily obtain that the hv-curvature tensor of the Berwald-type connection associated with $\overline{\overline{h}}$ vanishes (see Lemmma 4.1). This means that $(M,\widetilde{E})$ is a Wagner manifold. □

6.2. Definition. A Finsler manifold (M,E) is said to be *conformal* to a Berwald (or a locally Minkowski) manifold if there is a conformal change $\widetilde{g}=\varphi g$ $(\varphi=\exp\circ\beta^v)$ such that $(M,\widetilde{E})$ is a Berwald (or a locally Minkowski) manifold.

6.3. Theorem. *A Finsler manifold is conformal to a Berwald manifold if and only if it is a Wagner manifold.*

Proof. Let us suppose that the Finsler manifold (M,E) is conformal to a Berwald manifold, i.e., there is a conformal change $\widetilde{g}=\varphi g$ $(\varphi=\exp\circ\beta^v)$ such that $(M,\widetilde{E})$ is a Berwald manifold. Since the Berwald manifolds are, in particular, Wagner manifolds (cf. Prop. 3.12), in view of Theorem 6.1, the conformal change $g=\frac{1}{\varphi}\widetilde{g}$ yields a Wagner manifold with respect to the Wagner connection induced by $-\frac{1}{2}\beta\in C^\infty(M)$.

Explicitly, the Wagner endomorphism $\overline{h}$ and the Barthel endomorphism $\widetilde{h}$ of the Berwald manifold $(M,\widetilde{E})$ are related as follows:

$$\overline{h}=\widetilde{h}+\frac{1}{2}d\beta^v\otimes C.$$

Conversely, let us suppose that (M,E) is a Wagner manifold with respect to $(\overline{D},\overline{h},\alpha)$. Then, in view of Theorem 6.1, the conformal change $\widetilde{g}=\varphi g$ $(\varphi:=\exp\circ\beta^v,\ \beta:=-2\alpha)$ yields a Wagner manifold whose Wagner connection is induced by the function $\frac{1}{2}\beta+\alpha=-\alpha+\alpha=0$. Therefore (cf. Prop. 3.12) $(M,\widetilde{E})$ is a

Berwald manifold. The Barthel endomorphism $\widetilde{h}$ and the Wagner endomorphism $\overline{h}$ of the Wagner manifold (M, E) are related as follows:

$$\widetilde{h} = \overline{h} + d\alpha^v \otimes C. \qquad \square$$

6.4. Theorem. *A Finsler manifold is conformal to a locally Minkowski manifold if and only if it is a Wagner manifold and one (therefore all) of the conditions*

$$\text{(a)} \quad \overline{\Omega} = 0, \qquad \text{(b)} \quad \overline{\mathbb{R}} = 0, \qquad \text{(c)} \quad \overset{\circ}{\overline{\mathbb{R}}} = 0$$

are satisfied.

Proof. Let us suppose that the Finsler manifold (M, E) is conformal to a locally Minkowski manifold, i.e., there is a conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \beta^v$) such that $(M, \widetilde{E})$ is a locally Minkowski manifold. Then, in view of Theorem 6.3, (M, E) is a Wagner manifold with respect to the Wagner connection induced by the function $-\frac{1}{2}\beta \in C^\infty(M)$ and

$$\overline{h} = \widetilde{h} + \frac{1}{2} d\beta^v \otimes C.$$

Since $(M, \widetilde{E})$ is a locally Minkowski manifold, $\widetilde{\Omega} := -\frac{1}{2}[\widetilde{h}, \widetilde{h}] = 0$. This implies by an easy (but little lengthy) calculation, that $\overline{\Omega} := -\frac{1}{2}[\overline{h}, \overline{h}]$ also vanishes.

Conversely, if (M, E) is a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$ then, in view of Theorem 6.3, the conformal change $\widetilde{g} = \varphi g(\varphi := \exp \circ \beta^v$, $\beta := -2\alpha)$ yields a Berwald manifold with the Barthel endomorphism $\widetilde{h}$ such that

$$\widetilde{h} = \overline{h} + d\alpha^v \otimes C.$$

Now applying the further condition $\overline{\Omega} = 0$, we get:

$$\begin{aligned} \widetilde{\Omega} := -\frac{1}{2}[\widetilde{h}, \widetilde{h}] &= -\frac{1}{2}\big([\overline{h}, \overline{h}] + 2[\overline{h}, d\alpha^v \otimes C] + \\ &\quad + [d\alpha^v \otimes C, d\alpha^v \otimes C]\big) = -[\overline{h}, d\alpha^v \otimes C] - \frac{1}{2}[d\alpha^v \otimes C, d\alpha^v \otimes C] = \\ &\overset{\mathrm{I.(1.6i)}}{=} -d_{\overline{h}} d\alpha^v \otimes C + d\alpha^v \wedge [\overline{h}, C] - \frac{1}{2} d_{d\alpha^v \otimes C} d\alpha^v \otimes C + \\ &\quad + \frac{1}{2} d\alpha^v \wedge [d\alpha^v \otimes C, C] \overset{\mathrm{Cor.\ 3.4}}{=} \frac{1}{2} d\alpha^v \wedge [d\alpha^v \otimes C, C] \overset{\mathrm{I.(1.6h)}}{=} 0, \end{aligned}$$

since, for example,

$$d_{\overline{h}} d\alpha^v \overset{\mathrm{I.(1.6d)}}{=} -d d_{\overline{h}} \alpha^v = -d(d - d_{\overline{v}})\alpha^v \overset{\mathrm{I.(1.13c)}}{=} -dd\alpha^v = 0.$$

(An easy calculation shows that $d_{d\alpha^v \otimes C} d\alpha^v$ also vanishes.) $\square$

1. An observation on homogeneous functions

1.1. Remark. Let $k \in \mathbb{Z}$. We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *positive homogeneous* of degree k if for any vector $v \in \mathbb{R}^n \setminus \{0\}$ and positive real number t, we have

$$f(tv) = t^k f(v). \tag{1.1}$$

It is easy to check that if $f : \mathbb{R}^n \to \mathbb{R}$ is positive homogeneous of degree 0 and continuous at the point $0 \in \mathbb{R}^n$ then f is a constant function.

1.2. Proposition. *Let us select a subspace W of dimension $n-1$ and a nonzero vector q of $\mathbb{R}^n$ $(n \geq 2)$ such that*

$$\mathbb{R}^n = W \oplus \{tq \mid t \in \mathbb{R}\} =: W \oplus \mathcal{L}(q).$$

Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ has the following properties:

(i) *it is positive homogeneous of degree 0;*

(ii) *it is continuous at the points q, $-q$;*

(iii) *for any point $a \in W \setminus \{0\}$ and scalar $t \in \mathbb{R}$*

$$f(a + tq) = f(a).$$

Then f is constant on $\mathbb{R}^n \setminus \{0\}$.

Proof. Consider the function $f_1 := f \restriction W \setminus \{0\}$. Let $c : \mathbb{N} \to W \setminus \{0\}$, $n \to c_n$ be a sequence such that

$$\lim_{n\to\infty} c_n = 0.$$

Then

$$\lim_{n\to\infty} f_1(c_n) = \lim_{n\to\infty} f(c_n) \overset{\text{(iii)}}{=} \lim_{n\to\infty} f(c_n + q) = f(q),$$

since f is continuous at the point $q \in \mathbb{R}^n \setminus \{0\}$. This means that $f(q)$ is the limit of the function f_1 at $0 \in W$ and, consequently, the extended function

$$\widetilde{f}_1 : W \to \mathbb{R}, \quad a \to \widetilde{f}_1(a) := \begin{cases} f_1(a) & (a \neq 0) \\ f(q) & (a = 0) \end{cases}$$

is continuous at the point $0 \in W$ and it preserves the homogeneity property of the function f. Therefore, by Remark 1.1, $\widetilde{f}_1$ is constant and in any point $a \in W \setminus \{0\}$,

$$f(a) = \widetilde{f}_1(a) = \widetilde{f}_1(0) = f(q). \tag{1.2a}$$

Using the relation (1.2a), with the choice $b = a + tq$, where $a \in W \setminus \{0\}$, $t \in \mathbb{R}$, we have

$$f(b) = f(a + tq) \overset{\text{(iii)}}{=} f(a) \overset{\text{(1.2a)}}{=} f(q). \tag{1.2b}$$

To end the proof, it is enough to check that

$$f(q) = f(-q). \tag{1.2c}$$

This is almost trivial:

$$f(q) = \lim_{n\to\infty} f_1(c_n) = \lim_{n\to\infty} f(c_n) \overset{\text{(iii)}}{=} \lim_{n\to\infty} f(c_n - q) = f(-q),$$

since f is continuous at the point $-q \in \mathbb{R}^n \setminus \{0\}$. $\square$

2. $\mathcal{C}$-conformal changes of Riemann-Finsler metrics

2.1. Definition. Consider a Finsler manifold (M, E). A conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$, $\alpha \in C^\infty(M)$) is said to be *$\mathcal{C}$-conformal* at a point $p \in M$ if the following conditions are satisfied:

(2.1a) $\quad (d\alpha)_p \neq 0$, i.e., α is regular at the point p;

(2.1b) $\quad [J, \operatorname{grad} \alpha^v] = 0.$

2.2. Proposition. *Let (M, E) be a Finsler manifold and $\alpha \in C^\infty(M)$. Then the following assertions are equivalent:*

(a) $[J, \operatorname{grad} \alpha^v] = 0$.

(b) $i_{F \operatorname{grad} \alpha^v} \mathcal{C} = 0$.

(c) $\operatorname{grad} \alpha^v$ *is a vertical lift, i.e., there exists a vector field $X \in \mathfrak{X}(M)$ such that*

$$\operatorname{grad} \alpha^v = X^v. \tag{2.2}$$

Proof. (a) $\iff$ (b) It is an immediate consequence of II.(1.2d). Moreover, it can be seen from its proof that

$$\forall Y \in \mathfrak{X}(M) : [J, \operatorname{grad} \alpha^v] Y^c = 0 \iff [Y^v, \operatorname{grad} \alpha^v] = 0,$$

which implies by I.1.14 the equivalence (a) $\iff$ (c); hence our assertion. $\square$

2.3. Lemma and definition. *Consider a Finsler manifold (M, E) and let us suppose that the change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$, $\alpha \in C^\infty(M)$) is $\mathcal{C}$-conformal at a point $p \in M$. Let $\sigma \in \mathfrak{X}(M)$ be an arbitrary vector field with the property $\sigma(p) \neq 0$ which obviously implies that σ is nonvanishing over a connected open neighbourhood U of p. Then the mapping*

$$\langle\, , \rangle : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U), \tag{2.3a}$$

$$(Y, Z) \to \langle\, , \rangle(Y, Z) =: \langle Y, Z \rangle := g(Y^v, Z^v) \circ \sigma$$

is a (pseudo-) Riemannian metric. This metric is called the osculating Riemannian metric *along σ.*

If, in addition, $\operatorname{grad}_U \alpha \in \mathfrak{X}(U)$ *is the gradient of the function α with respect to $\langle\, , \rangle$ then*

$$(\operatorname{grad}_U \alpha)^v = \operatorname{grad} \alpha^v. \tag{2.3b}$$

Proof. Let $X \in \mathfrak{X}(M)$ be the vector field determined by the formula (2.2). Then for any vector field $Y \in \mathfrak{X}(U)$,

$$\begin{aligned} \langle X, Y \rangle &:= g(X^v, Y^v) \circ \sigma \overset{(2.2)}{=} g(\operatorname{grad} \alpha^v, Y^v) \circ \sigma = \\ &= \omega(\operatorname{grad} \alpha^v, Y^h) \circ \sigma = (Y^h \alpha^v) \circ \sigma = \\ &= (Y\alpha)^v \circ \sigma = (Y\alpha) \circ \pi \circ \sigma = Y\alpha, \end{aligned}$$

hence $X = \operatorname{grad}_U \alpha$ and, consequently,

$$(\operatorname{grad}_U \alpha)^v = \operatorname{grad} \alpha^v. \qquad \square$$

2.4. Remark. In the sequel we shall fix the vector field X determined by the formula (2.2) as σ in Lemma 2.3. (Note that the regularity property (2.1a) implies that $X(p) \neq 0$.)

Therefore, the osculating Riemannian metric $\langle\ ,\ \rangle$ will be considered as a mapping

$$\langle\ ,\ \rangle : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U), \tag{2.4}$$

$$(Y, Z) \to \langle\ ,\ \rangle(Y, Z) =: \langle Y, Z\rangle := g(Y^v, Z^v) \circ X,$$

where U is a fixed connected open neighbourhood of the point p such that for any $q \in U$, $X(q) \neq 0$.

2.5. Proposition. *Consider a Finsler manifold (M, E) with the Riemann-Finsler metric g and let us suppose that the change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$, $\alpha \in C^\infty(M)$) is $\mathcal{C}$-conformal at a point $p \in M$. If $W \subset TpM$ is a subspace of dimension $n-1$ such that $TpM = W \oplus \mathcal{L}(X(p))$ then for any tangent vector $w \in W \setminus \{0\}$ and $t \in \mathbb{R}$,*

$$g(Y^v, Z^v)(w + tX(p)) = g(Y^v, Z^v)(w).$$

Consequently, for any vector fields $Y, Z \in \mathfrak{X}(M)$, the function $g(Y^v, Z^v)$ is constant on $TpM \setminus \{0\}$.

Proof. For the sake of brevity, consider the parametric line

$$\ell : t \in \mathbb{R} \to \ell(t) := w + tX(p) \in TpM,$$

where $w \in W \setminus \{0\}$ is an arbitrary fixed tangent vector. Now let us define a function Θ as follows:

$$\Theta : t \in \mathbb{R} \to \Theta(t) := g(Y^v, Z^v) \circ \ell(t) \in \mathbb{R}.$$

If $(\pi^{-1}(U), (x^i, y^i)_{i=1}^n)$ is the chart induced by a chart $(U, (u^i)_{i=1}^n)$ on M then we have

$$\Theta'(t) = \left(\frac{\partial}{\partial x^i}\right)_{\ell(t)} (g(Y^v, Z^v)) \cdot (x^i \circ \ell)'(t) + \left(\frac{\partial}{\partial y^i}\right)_{\ell(t)} (g(Y^v, Z^v)) \cdot (y^i \circ \ell)'(t).$$

Here, for any $i \in \{1, \ldots, n\}$ and $t \in \mathbb{R}$,

$$x^i \circ \ell(t) = u^i \circ \pi \circ \ell(t) = u^i \circ \pi(w + tX(p)) = u^i(p),$$

i.e., $x^i \circ \ell$ is constant, and so for any $t \in \mathbb{R}$, $(x^i \circ \ell)'(t) = 0$.

On the other hand

$$y^i \circ \ell(t) = y^i(w + tX(p)) = w^i + tX^i(p),$$

therefore

$$\begin{aligned}
\Theta'(t) &= \left(\frac{\partial}{\partial y^i}\right)_{\ell(t)} (g(Y^v, Z^v)) \cdot (y^i \circ \ell)'(t) = X^i(p) \cdot \left(\frac{\partial}{\partial y^i}\right)_{\ell(t)} (g(Y^v, Z^v)) = \\
&= X^i \circ \pi(\ell(t)) \left(\frac{\partial}{\partial y^i}\right)_{\ell(t)} (g(Y^v, Z^v)) = (X^v g(Y^v, Z^v)) \circ \ell(t) = \\
&\overset{\mathrm{I.(3.3b),I.(3.6a)}}{=} 2\mathcal{C}_\flat(X^h, Y^h, Z^h) \circ \ell(t) = 2\mathcal{C}_\flat(FX^v, Y^h, Z^h) \circ \ell(t) = \\
&= 2\mathcal{C}_\flat(F \operatorname{grad} \alpha^v, Y^h, Z^h) \circ \ell(t) \overset{\text{Prop. 2.2}}{=} 0
\end{aligned}$$

and, consequently, Θ is also a constant function.

Thus for any real number $t \in \mathbb{R}$,

$$\Theta(t) = \Theta(0) :\Longleftrightarrow g(Y^v, Z^v)(w + tX(p)) = g(Y^v, Z^v)(w).$$

According to Proposition 1.2, this means that the function $g(Y^v, Z^v)$ is constant on $TpM \setminus \{0\}$, namely for any tangent vector $v \in TpM \setminus \{0\}$,

$$g(Y^v, Z^v)(v) = g(Y^v, Z^v)(X(p)) = \langle Y, Z\rangle(p). \qquad \square$$

2.6. Theorem. *Let (M, E) be a Finsler manifold. If there exists a $\mathcal{C}$-conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$, $\alpha \in C^\infty(M)$) at a point $p \in M$, then (M, E) is locally Riemannian, more precisely,*

$$g(Y^v, Z^v) = \langle Y, Z\rangle \circ \pi = \langle Y, Z\rangle^v,$$

where $\langle\, ,\, \rangle$ is the osculating Riemannian metric defined over U.

Proof. It is enough to mention that if $\widetilde{g} = \varphi g$ is a $\mathcal{C}$-conformal change at the point $p \in M$ then it is also such a change for any point $q \in U$. (Note that the assumption $X(q) \neq 0$ implies the regularity property $(d\alpha)_q \neq 0$ for any point $q \in U$.)

Therefore, the theorem is a direct consequence of Proposition 2.5. $\square$

2.7. Remarks. (a) Without loss of generality we can obviously assume that $\alpha(p) = 0$ under a $\mathcal{C}$-conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$, $\alpha \in C^\infty(M)$) at the point p. If, in addition, the Finsler manifold (M, E) is positive definite, then it is natural to consider the tangent space TpN, $N := \alpha^{-1}(0)$, as the subspace W in Proposition 2.5.

(b) Our result can be interpreted in case of a vector space endowed with a so-called Minkowski functional; cf. [31]. Under such an interpretation Proposition 2.5 states a new condition for Minkowski spaces to be Euclidean; but we omit the details.

(c) Note that Theorem 2.6 is based on the usual, but a relatively “rigid” definition of Finsler manifolds: the differentiability of the energy function is required at *all* nonzero tangent vectors; i.e. there is *no* singularity except from the zero vectors of tangent spaces. Actually, the main points are the homogeneity and continuity of the Riemann – Finsler metric along the gradient vector field of the scale function, which depends only on the “position” in case of a $\mathcal{C}$-conformal change. It can be easily seen that the following examples due to M. Hashiguchi are *not* within the competence of our result.

“When I wrote my thesis . . . , I imaged the following example as a non-Riemannian Finsler metric L admitting a $\mathcal{C}$-conformal change: Let m be a fixed integer such that $1 < m < n$. Indices a, b and λ, μ are supposed to take the values $1, \ldots, m$ and $m + 1, \ldots, n$, respectively. On $\mathbb{R}^n$ we consider L given by

$$L^2(x^i, y^i) = L_1^2(x^i, y^a) + L_2^2(x^i, y^\lambda),$$

where L_1 is a non-Riemannian Finsler metric, and L_2 is a Riemannian metric . . . . Especially, the three-dimensional Finsler metric L on $\mathbb{R}^3$ given by

$$L^2(x^1, x^2, x^3, y^1, y^2, y^3) = x^3 \frac{(y^1)^4}{(y^2)^2} + (y^3)^2$$

admits a $\mathcal{C}$-conformal change $\overline{L} = e^{\alpha}L$, where $\alpha := -\frac{1}{2}(x^3)^2$, which gives a so-called concurrent vector field α_i ". (Hashiguchi's letter to the author; 2000–01–05).

Indeed, a routine calculation shows that for example the gradient of the function $\alpha^v := \alpha \circ \pi$, $\alpha := -\frac{1}{2}(x^3)^2$ is just the vector field $\left(-x^3\frac{\partial}{\partial x^3}\right)^v = -(x^3 \circ \pi)\frac{\partial}{\partial y^3}$. Therefore, the elements of the matrix

$$(g_{ij})_{3\times 3} = x^3 \begin{pmatrix} 6\left(\dfrac{y^1}{y^2}\right)^2 & -4\left(\dfrac{y^1}{y^2}\right)^3 & 0 \\ -4\left(\dfrac{y^1}{y^2}\right)^3 & 3\left(\dfrac{y^1}{y^2}\right)^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are neither continuous along the vector field $X := -x^3\frac{\partial}{\partial x^3}$ and even nor defined there.

IV. SUMMARY

In this Ph.D. dissertation first of all we undertake a quite comprehensive survey of general theoretical elements of Finsler geometry. The primary aim of this survey is to present a standard system of notations and terminology built on three pillars: the theory of horizontal endomorphisms, the calculus of vector-valued forms and a "tangent-bundle version" of the method of moving frames. On the other hand we present a systematic treatment of some distinguished Finsler connections and some special Finsler manifolds. In particular we are interested in the conformal theory of Riemann-Finsler metrics and the theory of Wagner connections and Wagner manifolds. As we shall see, they are closely related. Finally, we investigate a special conformal change of the metric proving that its existence implies the Finsler manifold to be Riemannian. (The necessity is clear.)

I

This dissertation is divided into three parts. In part I first of all we present a quite detailed exposition of the conceptual and calculational background. Our main purpose is to insert the theory of Finsler connections and the foundations of special Finsler manifolds into a new approach of Finsler geometry. The first epoch-making steps in this direction were done by J. Grifone [10], [11], our work can be considered as a systematic continuation of the program initiated by him. Following Grifone's theory of nonlinear connections (whose role is played in our presentation by the so-called *horizontal endomorphisms*) we use systematically an "intrinsic" calculus based on the Frölicher-Nijenhuis formalism. Technically, we enlarged and – at the same time – simplified the apparatus by using the tools of tangent bundle differential geometry. This means first of all the consistent use of a special frame field, constituted by vertically and completely (or vertically and horizontally) lifted vector fields. Thus the third pillar of our approach is the method of moving frames. It has a decisive superiority in calculations over coordinate methods: the formulation of the concepts and results becomes perfectly transparent, and the proofs have a purely intrinsic character. We believe that the compact, elegant and efficient formulation presented here demonstrate the power of our approach. For example, in section I/4 we present an invariant and axiomatic description of three notable Finsler connection (linear connections associated to a nonlinear one with the help of some conditions of compatibility): the *Berwald, Cartan* and *Chern-Rund connections*. Theorems are organized as follows: the first group of axioms characterizes a unique Finsler connection allowing us to derive the explicit rules of calculations for the corresponding covariant derivatives. Adding further conditions to them, the second group yields the characterization of the three classical Finsler connections. Although these results belong to the foundations they are new. Moreover, we hope that they help in better understanding the role of the different axioms, and open a path for further, essential generalizations. As motivations, we can mention the so-called *Wagner connections* (as *generalized Cartan connections*, i.e. Cartan connections with nonvanishing $(h)h$-torsion) or the associated *Berwald-type* Finsler connections (as *generalized Berwald connections*, i.e. Berwald connections with nonvanishing $(h)h$-torsion).

II

In part II we start with the definition of conformal equivalence of Riemann-Finsler metrics. (This relation is formally the same as that in Riemannian geometry.) We give a modern proof of Knebelman's famous observation which points out that the scale function between conformally equivalent Riemann-Finsler metrics must be independent of the "direction", i.e. it is a vertical lift. We also derive some important conformal invariants and transformation formulas. As an application of the results a well-known classical theorem will be proved intrinsically. It states (in H. Weil's terminology; [30], p. 226) that "*the projective and conformal properties of a Finsler space determine its metric properties uniquely*".

In this part we also demonstrate that the Frölicher-Nijenhuis formalism provides a perfectly adequate conceptual and technical framework for the study even of such complicated objects as Wagner connections. Our intrinsically formulated and proved results not only cover the classical local results but give a much more precise and transparent picture and open new perspectives. First of all we establish an explicit formula between the (canonical) Barthel endomorphism and a Wagner endomorphism (the nonlinear part of a Wagner connection). Then we calculate its tension, weak and strong torsion, i.e. data determining uniquely a nonlinear connection by Grifone's theory. It turns out that the rules of calculation with respect to a Wagner connection are formally the same as those with respect to the classical Cartan connection. These investigations are based on a number of some new (but more or less) technical observations and a fine analysis of the second Cartan tensor belonging to a Wagner endomorphism. Using these results an important classical theorem on the so-called *Landsberg manifolds*, first formulated and proved intrinsically by J. G. Diaz will be generalized. The classical version contains equivalent characterizations of the vanishing of the second Cartan tensor belonging to the Barthel endomorphism (i.e. the canonical nonlinear connection of the Finsler manifold). In his thesis [8] the author gives a coordinate-free proof of the theorem using several explicit relations between the classical Cartan tensors and curvatures (or their lowered tensors) of the Cartan connection. We managed to reduce the number of these relations to some of fundamental ones and the theorem is proved in generality of Wagner connections and Wagner manifolds. *Techniques we need to discuss them are suitable to reproduce lots of classical results as well.* We found this observation very useful.

Finally, after a new intrinsic definition as well as several tensorial characterizations of Wagner manifolds we present coordinate-free proofs of Hashiguchi-Ichijyō's theorems to clarify the geometrical meaning of this special class of Finsler manifolds. In the classical terminology: "The condition that a Finsler space be conformal to a Berwald space is that the space becomes a Wagner space with respect to a gradient $\alpha_i(x)$", (see [16], Theorem B).

III

In part III we deal with a special conformal change of Riemann-Finsler metrics introduced by M. Hashiguchi [14]. The point of the *C-conformality* is that we require the vanishing of one of conformal invariants. Under this hypothesis the gradient vector field of the scale function becomes independent of the "direction", i.e. it will be a vertically lifted vector field. (Vector fields with such a property is called *concurrent* too; see e.g. [14], [28] and [37].)

In his cited work [14] Hashiguchi proved for some special Finsler manifolds (in his terminology: two-dimensional spaces, C-reducible spaces, spaces with (α, β)-metric etc.) that the existence of a C-conformal change of the metric implies that the manifold is Riemannian (at least locally). Here we show that Hashiguchi's result is valid without any extra condition. In terms of our characterization this means that the vanishing of some conformal invariants, like the conformal invariant first Cartan tensor, can be interpreted as a sufficient condition for a Finsler manifold to be Riemannian. (The necessity is clear.) Our result is based on a usual, but relatively "rigid" definition of Finsler manifolds: the differentiability is required at *all* nonzero tangent vectors, i.e. there is *no* singularities except for the zero vectors of tangent spaces. Actually, the main points are the homogeneity and continuity of the Riemann-Finsler metric along the gradient vector field of the scale function which depends only on the "position" in case of a C-conformal change. Weakening the condition of differentiability new perspectives open to investigate the C-conformality. As an illustration we shall cite some valuable fragments from Hashiguchi's original ideas in one of the last remarks.

V. ÖSSZEFOGLALÓ

Disszertációnkban mindenekelőtt egy meglehetősen részletes áttekintését adjuk a Finsler-geometria általános elméleti alapjainak. Tesszük ezt egyfelől azért, mert a Finsler-geometriában talán a mai napig sem alakult ki az a széleskörű terminológiai, jelölésbeli stb. konszenzusrendszer, mely egyértelműen feleslegessé tenné a hosszasabb előkészületeket. Célunk tehát a jelölés- és szóhasználat rögzítése. Ezen túlmenően, disszertációnkban a Finsler-geometria (Finsler-konnexiók elmélete, speciális Finsler-sokaságok stb.) koordinátamentes formában kerül bemutatásra, a klasszikus tenzorkalkulus egy jelentősen továbbfejlesztett változatának kalkulatív keretei között. Megközelítésünkben alapvető szerepet játszik a nemlineáris konnexiók Grifone-féle elmélete, szoros összefüggésben a vektorértékű differenciálformák A. Frölicher és A. Nijenhuis [9] által kidolgozott kalkulusával. Segítségükkel a Finsler-konnexiók és a speciális Finsler-sokaságok szisztematikus, egységes tárgyalására nyílik lehetőség. Külön fejezetet szenteltünk a konform ekvivalens Riemann-Finsler-metrikák, a Wagner-konnexiók (mint speciális Finsler-konnexiók) és a Wagner-sokaságok (mint speciális Finsler-sokaságok) elméletének; látni fogjuk, hogy ezek a témakörök természetes módon kapcsolódnak egymáshoz. A harmadik fejezetben a konform reláció egy speciális esetével, a Riemann-Finsler-metrika un. *C-konform* ,,változtatásaival" (change) foglalkozunk. Bizonyítást nyer, hogy amenynyiben ez lehetséges, úgy a szóban forgó Finsler-sokaság Riemann-sokaságra redukálódik: a Riemann-Finsler-metrika egy az alapsokaságon adott Riemann-metrika vertikális liftje. (A szükségesség nyilvánvaló.)

I

A Finsler-geometriai problémák általunk követett koordinátamentes tárgyalásának alapjait illetően kivételes egyértelműséggel utalhatunk J. Grifone nevezetes [10] és [11] dolgozataira, jóllehet a nemlineáris konnexiók Grifone-féle elméletén belül a Finsler-konnexiók ,,speciális esetként" jelentkeznek. (Arról van szó ugyanis, hogy a Finsler-sokaságok kanonikus nemlineáris konnexiója, az un. *Barthel-endomorfizmus* ,,metrikus", azaz speciális módon származik a szóban forgó Finsler-sokaság alapfüggvényéből – ez a Finsler-geometria alaplemmája, amely analogonja a Riemann-sokaságok Lévi-Civita konnexiójának egzisztenciáját állító nevezetes eredménynek.) A kezdeteket és a folytatást illetően, természetesen a teljesség igénye nélkül, elsősorban H. Akbar-Zadeh, P. Dazord, J. G. Diaz, N. L. Youssef és M. Crampin, a hazai geométerek közül pedig J. Szilasi munkáit emelhetjük ki; utóbbiak, különös tekintettel a [36] dolgozatra, szerves egységet alkotnak disszertációnk bevezető részével. Témaválasztásunkat az általuk megkezdett munka következetes továbbvitele, a fentiekben körvonalazott megközelítés és módszerek érvényességi körének kiterjesztése motiválta: a rendelkezésünkre álló kereteken belül kísérletet teszünk a Riemann-Finsler metrikák konform elméletének modern feldolgozására. Az apparátust egyszerűsíti – egyszersmind gyorsabbá és hatékonyabbá teszi az érintősokaságon konstruált (vektormezők vertikális és teljes, ill. vertikális és horizontális liftjei által alkotott) „frame"-ek alkalmazása; ezzel az eszközzel sűrűn fogunk élni meggondolásaink során. Ilyen körülmények között nem támaszkodhattunk kielégítőképpen a témakör napjainkban is szinte egyeduralkodó [24], [30] klasszikus monográfiáira, vezérvonalként és inspiráló hatásuk miatt azonban nem is nélkülözhettük őket. (Kézenfekvő példa a Finsler-konnexiók Matsumoto-féle elmélete, mely kiindulópontként egy az érintőnyalábhoz csatolt kisérőél-nyaláb segítségével konstruált principális nyalábot vesz alapul, az un. *Finsler-nyalábokat*. Ezeknek a nyalábοknak a konnexiói – durván szólva a vertikális disztribúció direkt komplementerei – adják a kulcsot a Finsler-konnexiók fogalmának értelmezéséhez: „Differential-geometric objects and quantities are mainly introduced in the principal bundle over the tangent bundle induced from the linear frame bundle ... " ([24], 46. old.). Mindenesetre itt a Finsler-konnexiók rendszerező, szisztematikus feldolgozásáról van szó.) Arra törekedtünk tehát, hogy az olvasó minél teljesebb képet kapjon a tárgyról, s munkánk – amennyire ez lehetséges – önmagában megállhasson.

Mint jeleztük, disszertációnk első részében rögzítjük a szó- és jelöléshasználatot. Ezen túlmenően a Finsler-geometria alapjait tekintjük át. Külön figyelmet szenteltünk az un. első és – kiváltképp – a második Cartan-tenzoroknak, melyek nélkülözhetetlenek már a legalapvetőbb összefüggések és geometriai konstrukciók esetén is. Minthogy a második Cartan-tenzor a szokásosnál jóval általánosabb szituációban kerül bevezetésre – a kanonikus Barthel-endomorfizmus helyett tetszőleges „nemlineáris konnexiót" (horizontális endomorfizmust) veszünk alapul-, részletesen vizsgáljuk szimmetriatulajdonságainak kapcsolatát a szóban forgó horizontális endomorfizmus jellemző adataival. A kapott eredmények – bár kifejezetten az alapokhoz tartoznak – újak. Megközelítésünk előnyei részben a Wagner-konnexiók és a Wagner-sokaságok tárgyalásánál jelentkeznek, részben pedig a nevezetes Finsler-konnexiók leírásánál. Az utóbbit illetően független axiómák megadására törekedtünk és igyekeztünk rávilágítani geometriai jelentésükre. Nevezetesen, az axiómák első csoportja mindíg egy Finsler-konnexiót karakterizál és lehetővé teszi a kovariáns deriválás formuláinak explicit megadását. A második csoportban szereplő axiómák „finomítják" a konstrukciót és segítségükkel visszakapjuk a klaszszikus Berwald-, Cartan- és Chern-Rund-konnexiókat. Az első rész lezárásaként a későbbiekben fontos szerepet játszó speciális Finsler-sokaságokkal, a Berwald-, illetve lokálisan Minkowski-sokaságokkal foglalkozunk, élve a választott eszközök és keretek nyújtotta elegáns tárgyalás lehetőségével. Így – noha az áttekintett tények jó része klasszikus – a közölt bizonyítások kivétel nélkül eredetiek.

II

Tekintsünk egy (M, E) Finsler-sokaságot. Ismeretes, hogy az

$$E : TM \to R,$$

un. *energiafüggvény*, mely bizonyos simasági, homogenitási és regularitási feltételeknek tesz eleget, egy (pseudo-) Riemann-metrikát származtat a TTM „érintőnyaláb” vertikális résznyalábján; a

$$g : v \in TM \setminus \{0\} \to g_v, \quad g_v : T^v{}_vTM \times T^v{}_vTM \to R$$

un. *Riemann-Finsler-metrikát.* (Egy újabb dolgozatban a *vertikális metrika* elnevezés is használatos, ami rövidebb és kifejező; ld. [34].) A szóban forgó metrikák közötti konform kapcsolat közvetlen analogonja a Riemann-sokaságok elméletében is fontos szerepet játszó

$$\widetilde{g} = \varphi g$$

relációnak, ahol φ pozitív értékű, s a nullmetszettől eltekintve sima függvény a TM érintősokaságon. Tekintettel a Finsler-geometriában szokásos módon előírt homogenitási feltételekre, következményként adódik, hogy konform ekvivalens Riemann-Finsler-metrikák csupán egy „iránytól” független, azaz érintőterenként konstans függvényszorzóban különböznek. Így az un. „skálázó” függvény a teljes érintőtéren sima függvénnyé terjeszthető ki, pontosabban mindig felírható

$$\varphi = \exp \circ \alpha \circ \pi$$

alakban, ahol α az alapsokaságon adott sima függvény, $\pi : TM \to M$ pedig a természetes projekció. Ez a nevezetes észrevétel M. S. Knebelman [19] nevéhez fűződik, s modern eszközökkel bizonyítjuk a II/1 szakaszban. Itt és a II/2 szakaszban néhány a továbbiak szempontjából nélkülözhetetlen konform-invariáns tenzorral és transzformációs formulával is foglalkozunk. Mindenekelőtt explicit relációt vezetünk le a konform ekvivalens Riemann-Finsler metrikákhoz tartozó kanonikus sprayk kapcsolatára; azonnali következményként adódik a Barthel-endomorfizmusok közötti kapcsolat. Ezek birtokában – elvileg – bármely más kanonikus objektum és operáció (a Berwald-, illetve a Cartan-konnexióra vonatkozó kovariáns deriválás, görbületi tenzorok stb.) „változása” leírható – több-kevesebb gyakorlati nehézség árán. Illusztrációképpen a Barthel-endomorfizmushoz csatolt második Cartan-tenzorra vonatkozó transzformációs formulát vezetjük le. (A részletes áttekintést illetően – tenzorkomponensek és konnexió-paraméterek nyelvén – M. Hashiguchi klasszikus [14] dolgozatára utalhatunk.) A nyert eredmények alkalmazásaként modern eszközökkel bizonyítjuk H. Weyl klasszikus tételét az egyidejűleg konform és projektív kapcsolatban álló Finsler-sokaságokról. Ekkor a Riemann-Finsler metrikák csupán egy konstans szorzóban különböznek egymástól. Szemléletesen szólva, egy Finsler-sokaság konform és projektív jellemzői (properties) egyértelműen meghatározzák a metrikus viszonyokat (metric properties; v.ö. [30], 226. old.) A II/3 szakaszban rátérünk a Wagner konnexiók tárgyalására. Ezeket a speciális Finsler-konnexiókat elsőként V. V. Wagner vezette be és tette vizsgálat tárgyává [40] dolgozatában. Segítségükkel értelmezte az általánosított Berwald-sokaságok (speciális esetként az utóbb róla elnevezett *Wagner-sokaságok*) fogalmát és igazolta, hogy az un. ,,cubic", azaz köb-metrikával rendelkező kétdimenziós Finsler-sokaságok egyidejűleg általánosított Berwald-sokaságok is. Az elmélet magasabb dimenziós esetekre való kiterjesztését és alkalmazásait illetően elsősorban M. HASHIGUCHI (a Wagner-konnexiók axiomatikus leírása) és Y. ICHIJYŌ ért el számottevő eredményeket. Fény derült arra, hogy a Wagner-sokaságok osztálya zárt a Riemann-Finsler-metrika konform ,,változtatására" nézve, továbbá tisztázódott a szóban forgó Finsler-sokaságok viszonya a (klasszikus) Berwald-, illetve lokálisan Minkowski-sokaságokhoz:

Wagner-sokaságok (integrálható Wagner-endomorfizmussal)

$\nwarrow \quad \nearrow$

$\searrow \quad \swarrow$

A metrika konform változtatása

$\nearrow \quad \nwarrow$

$\swarrow \quad \searrow$

Berwald-sokaságok $\dashrightarrow$ Lokálisan Minkowski-sokaságok

Integrabilitási feltétel

Röviden szólva, minden konform változtatáshoz hozzárendelünk egy speciális Finsler-konnexiót, az un. *Wagner-konnexiót*. Ezek után a Wagner-sokaságok a klasszikus Berwald-sokaságok mintájára vezethetők be a Wagner-konnexiók nemlineáris része, az un. *Wagner-endomorfizmus* által meghatározott ,,Berwald-tipusú" Finsler-konnexió segítségével. (Analitikus nyelven ez ,,csupán" annyit jelent, hogy megköveteljük a Wagner-endomorfizmus simaságát a teljes érintősokaságon. Ekvivalens módon: a szóban forgó horizontális struktúrát – a Wagner-endomorfizmust – egy az alapsokaságon adott lineáris konnexió származtatja. Ekkor a csatolt Berwald-tipusú Finsler-konnexió egybeesik a lineáris konnexió horizontális liftjével.) Az alábbi, Hashiguchitól és Ichijyōtól származó szép tétel pedig rávilágít a Wagner-sokaságok geometriai jelentőségére: egy Finsler-sokaság pontosan akkor Wagner-sokaság, ha Riemann-Finsler-metrikája konform ekvivalens egy Berwald-sokaság Riemann-Finsler-metrikájával (ld. II/6 szakasz, Theorem 6.3; [16]).

Disszertációnk második része hivatott demonstrálni, hogy a Frölicher-Nijenhuis-formalizmus egy tökéletesen adekvát fogalmi és kalkulatív eszköznek bizonyul a Riemann-Finsler-metrikák konform elméletének haladottabb szintjén éppúgy, mint az alapokat illetően. Mindenekelőtt explicit formulát vezetünk le a kanonikus Barthel-endomorfizmus és a Wagner-konnexiókhoz tartozó Wagner-endomorfizmusok kapcsolatára. Ez lehetővé teszi, hogy meghatározzuk az utóbbi tenzióját, gyenge- és erős torzióját, vagyis azokat az adatokat, melyek a nemlineáris konnexiók Grifone-féle elmélete szerint egyértelműen meghatározzák az operációt. A Wagner-endomorfizmusok ,,metrikus" karakterével kapcsolatban részletesen megvizsgáljuk a hozzájuk csatolt második Cartan-tenzor tulajdonságait néhány új, többé-kevésbé technikai jellegű észrevétel felhasználásával. Mindezek birtokában lehetővé válik a Wagner-konnexiók szerinti kovariáns deriválás szabályainak explicit meghatározása, melyek a klasszikus Cartan-konnexió szerinti kovariáns deriválás közvetlen analogonjainak bizonyulnak. Alapvető görbületi azonosságokat (beleértve a Bianchiidentitásokat is) vezetünk le a II/4 szakaszban és segítségükkel Wagner sokaságokra általánosítjuk a Finsler-geometria egyik fontos tételét a ,,Landsberg sokaságok" jellemzéséről. Speciális esetként természetesen visszakapjuk a klasszikus eredményt; a Barthel-endomorfizmushoz csatolt második Cartan-tenzor eltűnésével ekvivalens tenzoriális feltételeket. Ezek koordinátamentes megfogalmazása és bizonyítása megtalálható J. G. Diaz hasznos ösztönzéseket adó [8] munkájában. Ebben a szerző számos olyan explicit formulát használ fel, melyek a klasszikus Cartan-konnexió görbületi adatait kapcsolják össze a Cartan-tenzorokkal, illetve azok kovariáns deriváltjaival. Az általunk adott bizonyítás egyrészt a Wagner-konnexiók (Wagner-sokaságok) általánosságában mozog, másrészt pedig sikerült redukálni a szóban forgó explicit formulák számát néhány alapvető görbületi azonosságra. E fejezet lezárásaként a Wagner-sokaságok intrinsic értelmezését, illetve tenzoriális jellemzését követően az eddigiekhez adekvát módon fogalmazzuk meg és bizonyítjuk Hashiguchi és Ichijyō legfontosabb eredményeit a Wagner-sokaságokkal kapcsolatban.

III

Disszertációnk harmadik részében a Riemann-Finsler-metrikák konform relációjának egy M. Hashiguchi által a [14] dolgozatban bevezetett speciális esetével foglalkozunk. Értelmezésünkben az un. *C-konformalitást* a II/1 szakaszban leírt konform-invariáns tenzorok egyikének eltűnése jellemzi; Proposition 1.12. (Érdekességképpen megemlítjük, hogy az idézett helyen elsőként szereplő konform-invariáns tenzor (v.ö. Proposition 1.9) eltűnését kizárják a Finsler-sokaságok energiafüggvényével kapcsolatban előírt homogenitási feltételek, míg az ugyancsak konform-invariáns első Cartan-tenzor eltűnése jól ismert módon azt jelenti, hogy a Finsler-sokaság Riemann-sokaságra redukálódik. A kérdés tehát az, hogy milyen következményekkel jár egy a harmadik csoportba tartozó konform-invariánssal kapcsolatban előírt, analóg feltétel teljesülése.) Idézett dolgozatában Hashiguchi bebizonyította, hogy számos speciális Finsler-sokaság esetén (kétdimenziós Finsler-sokaságok, C-redukálható Finsler-sokaságok, Finsler-sokaságok (α, β)-metrikával stb.), a C-konform változtatást megengedő metrika valójában egy az alapsokaságon adott Riemann-metrika vertikális liftje, azaz a tekintett Finsler-sokaság Riemann-sokaságra redukálható. Az általunk elvégzett vizsgálatok azonban azt mutatják, hogy ugyanez teljesül *tetszőleges* Finsler-sokaság esetén, amennyiben a Finsler-sokaságok szokásos, de relatíve ,,merev" értelmezését vesszük alapul. Ez azt jelenti, hogy a nullmetszettől eltekintve mindenütt megfelelő simasági feltételeket írunk elő, vagyis kizárjuk a további szingularitásokat. A lényeges mozzanat a Riemann-Finsler-metrika homogenitása és – némi pontatlansággal – folytonossága a skálázó függvény csupán ,,helytől" függő gradiense mentén. (Az ilyen tulajdonságú vektormezőket *konkurrens vektormezőként* is emlegeti a szakirodalom; ld. pl. [14], [28] és [37].) Eredményünk alapján a szóban forgó konform-invariáns eltűnése – hasonlóan az első Cartan-tenzorhoz – elegendő feltétele a Finsler-sokaságok ,,Riemannizálhatóságának". (A szükségesség nyilvánvaló.) Gyengítve a Finsler-sokaságokkal kapcsolatban előírt feltételeken, a C-konformalitás további vizsgálata is lehetővé válik. Illusztrációképpen M. Hashiguchi gondolataiból idézünk a fejezetet lezáró megjegyzések egyikében.

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[36]J. Szilasi and Cs. Vincze, *A new look at Finsler connections and special Finsler manifolds*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **16** (2000), 33–36.

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VII. Appendix

1. List of publications

[1] J. Szilasi and Cs. Vincze, *On conformal equivalence of Riemann-Finsler metrics*, Publ. Math. Debrecen **52** (1–2) (1998), 167–185.

[2] Cs. Vincze, *On C-conformal changes of Riemann-Finsler metrics*, Rend. Circ. Mat. Palermo, II. Ser., Suppl. **59** (1999), 221–228.

[3] Cs. Vincze, *On Wagner connections and Wagner manifolds*, Acta Math. Hung., accepted for publication.

[4] Cs. Vincze, *An intrinsic version of Hashiguchi-Ichijyō's theorems for Wagner manifolds*, SUT J. Math., Vol. **35**, No. 2 (1999), 263–270.

[5] J. Szilasi and Cs. Vincze, *A new look at Finsler connections and special Finsler manifolds*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **16** (2000), 33–63.

[6] Cs. Vincze, *On the existence of C-conformal changes of Riemann-Finsler metrics*, Tsukuba J. Math., accepted for publication.

2. List of citations

The number in square brackets indicates the number of our paper (cf. List of publications), to which the relevant article refers.

[1] T. Aikou, *Some remarks on the conformal equivalence of complex Finsler structures*, Finslerian geometries (A meeting of minds) (by P.L. Antonelli, ed.), Kluwer Academic Publishers, Netherlands, 1999, pp. 35–52.

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[3] J. Szilasi, *Notable Finsler connections on a Finsler manifold*, Lect. Mat. **19** (1998), 7–34.

[3] J. Szilasi and Sz. Szakál, *A new approach to generalized Berwald manifolds II*, submitted.

[4] J. Szilasi and Sz. Szakál, *A new approach to generalized Berwald manifolds II*, submitted.

[5] J. Szilasi and Sz. Szakál, *A new approach to generalized Berwald manifolds I*, submitted.

[5] J. Szilasi and Sz. Szakál, *A new approach to generalized Berwald manifolds II*, submitted.

[5] V. Tóth, *On Randers manifolds*, Stud. Cercet. Mat., accepted for publication.

[5] Sz. Vattamány, *Landsberg manifolds with vanishing Douglas tensor*, Publ. Math. Debrecen, accepted for publication.

3. Participations in conferences

The 3rd International Workshop on Differential Geometry and its Applications (The First German-Romanian Seminar on Geometry), 18–23 September, 1997, Sibiu, Romania

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The 18th Winter School GEOMETRY and PHYSICS, 10–17 January, 1998, Srni, Czech Republic

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The 20th Winter School GEOMETRY and PHYSICS, 15–22 January, 2000, Srni, Czech Republic

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The 11th National Seminar on Finsler and Lagrange Geometry, 17–20 February, 2000, Bacau, Romania