Dr. Varga Adrienn

Mathematical Analysis
Exercises for Engineers

Debreceni Egyetem Műszaki Kar
Műszaki Alaptárgyi Tanszék

DEBRECENI EGYETEM
MŰSZAKI KAR
MŰSZAKI ALAPTÁRGYI TANSZÉK

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MATHEMATICAL ANALYSIS – EXERCISES FOR ENGINEERS

Debreceni Egyetemi Kiadó
Debrecen University Press
2019

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ISBN 978-963-318-815-6

Kiadta: a Debreceni Egyetemi Kiadó Debrecen University Press
Felelős kiadó: Karácsony Gyöngyi
Nyomdai munkálatokat
a Debreceni Egyetem sokszorosítóüzeme végezte 2019-ben
dupress.unideb.hu

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Preface

This exercise book contains the study material in mathematics for engineer students in connection with differentiation and integration. It is mostly intended to help the studies of engineer students at the University of Debrecen.

1

Differentiation for univariable functions

1.1 Differentiation rules

Rules of operations:

$$(c\cdot f)' = c\cdot f' \qquad (f+g)' = f'+g' \qquad (f\cdot g)' = f'\cdot g + f\cdot g'$$
$$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2} \qquad (f\circ g)' = f'\circ g\cdot g'$$

Derivative of some elementary functions:

$f(x) =$	c	x^α	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\arcsin(x)$	$\arccos(x)$
$f'(x) =$	0	$\alpha x^{\alpha-1}$	$\cos x$	$-\sin x$	$\frac{1}{\cos^2 x}$	$\frac{-1}{\sin^2 x}$	$\frac{1}{\sqrt{1-x^2}}$	$-\frac{1}{\sqrt{1-x^2}}$

$f(x) =$	$\arctan(x)$	$\operatorname{arccot}(x)$	e^x	a^x	$\ln x$	$\log_a x$	$\sinh x$	$\cosh x$
$f'(x) =$	$\frac{1}{1+x^2}$	$-\frac{1}{1+x^2}$	e^x	$a^x \ln a$	$\frac{1}{x}$	$\frac{1}{x\ln a}$	$\cosh x$	$\sinh x$

$f(x) =$	$\tanh x$	$\coth x$	$\operatorname{arsinh}(x)$	$\operatorname{arcosh}(x)$
$f'(x) =$	$\frac{1}{\cosh^2 x}$	$\frac{-1}{\sinh^2 x}$	$\frac{1}{\sqrt{x^2+1}}$	$\frac{1}{\sqrt{x^2-1}}$

$f(x) =$	$\text{artanh}(x)$ $(\lvert x\rvert < 1)$	$\text{arcoth}(x)$ $(\lvert x\rvert > 1)$
$f'(x) =$	$\frac{1}{1-x^2}$	$\frac{1}{1-x^2}$

Sometimes we use alternative notation for the functions bellow:

$$\ln x = \log_e x, \quad \tan x = \text{tg} x, \quad \cot x = \text{ctg} x, \quad \sinh x = \text{sh} x, \quad \cosh x = \text{ch} x$$

Exercise 1.1.1 *Differentiate the following functions:*

$$f(x) = \tfrac{1}{x^4} + 5 \cdot \sqrt[3]{x} + 10^5 - 8 \cdot 10^x$$
$$f(x) = \left(\tfrac{1}{x} - 7x\right) \cdot \text{arctg } x - \lg x \cdot 10^x$$
$$f(x) = \log_4 x + 5,8 \cdot \text{tg } x - \ln x + x - 1,45 \cdot \cos x$$
$$f(x) = \tfrac{\arcsin x + x}{1 - \text{ctg } x}$$

Solution. • $f(x) = \frac{1}{x^4} + 5 \cdot \sqrt[3]{x} + 10^5 - 8 \cdot 10^x$ Using the identities

$$x^{-n} = \frac{1}{x^n} \qquad \sqrt[n]{x^m} = x^{\frac{m}{n}}$$

we get that

$$f'(x) = \left(x^{-4} + 5 \cdot x^{\frac{1}{3}} + 10^5 - 8 \cdot 10^x\right)' =$$
$$= (x^{-4})' + 5 \cdot (x^{\frac{1}{3}})' + (10^5)' - 8 \cdot (10^x)' =$$
$$= -4x^{-5} + 5 \cdot \frac{1}{3} \cdot x^{-\frac{2}{3}} + 0 - 8 \cdot 10^x \cdot \ln 10 =$$
$$= -\frac{4}{x^5} + \frac{5}{3} \cdot \frac{1}{\sqrt[3]{x^2}} - 10^x \cdot \ln 10^8$$

• $f(x) = \log_4 x + 5,8 \cdot \text{tg } x - \ln x + x - 1,45 \cdot \cos x$

$$f'(x) = \frac{1}{x \cdot \ln 4} + 5,8 \cdot \frac{1}{\cos^2 x} - \frac{1}{x} + 1 + 1,45 \cdot \sin x$$

• $f(x) = \left(\frac{1}{x} - 7x\right) \cdot \text{arctg } x - \lg x \cdot 10^x$

$$f'(x) = \left((\frac{1}{x} - 7x) \cdot \text{arctg } x\right)' - (\lg x \cdot 10^x)' =$$

$$= (x^{-1} - 7x)' \cdot \text{arctg } x + (\frac{1}{x} - 7x) \cdot (\text{arctg } x)' - [(\lg x)' \cdot 10^x + \lg x \cdot (10^x)'] =$$

$$= (-x^{-2} - 7) \cdot \text{arctg } x + (\frac{1}{x} - 7x) \cdot \frac{1}{1+x^2} - \left[\frac{1}{x \cdot \ln 10} \cdot 10^x + \lg x \cdot 10^x \cdot \ln 10\right] =$$

$$= (-\frac{1}{x^2} - 7) \cdot \text{arctg } x + (\frac{1}{x} - 7x) \cdot \frac{1}{1+x^2} - \left[\frac{1}{x \cdot \ln 10} \cdot 10^x + \lg x \cdot 10^x \cdot \ln 10\right]$$

- $f(x) = \frac{\arcsin\ x + x}{1 - \text{ctg}\ x}$

$$f'(x) = \frac{(\arcsin\ x + x)' \cdot (1 - \text{ctg}\ x) - (\arcsin\ x + x) \cdot (1 - \text{ctg}\ x)'}{(1 - \text{ctg}\ x)^2} =$$

$$f'(x) = \frac{\left(\frac{1}{\sqrt{1-x^2}} + 1\right) \cdot (1 - \text{ctg}\ x) - (\arcsin\ x + x) \cdot \left(-\left(-\frac{1}{\sin^2\ x}\right)\right)}{(1 - \text{ctg}\ x)^2} =$$

$$f'(x) = \frac{\left(\frac{1}{\sqrt{1-x^2}} + 1\right) \cdot (1 - \text{ctg}\ x) - \frac{\arcsin\ x + x}{\sin^2\ x}}{(1 - \text{ctg}\ x)^2}$$

Exercise 1.1.2 *Differentiate the following functions:*

$$f(x) = (4x-2)^6 + \sqrt[3]{x - x^7}$$
$$f(x) = \text{tg}(-x) + \text{arctg}(1 - x^2)$$
$$f(x) = 6^{4x-2} + \frac{\sin(13x^2)}{\text{tg}\left(\frac{x}{5}\right)}$$
$$f(x) = \cos^3(\sqrt[10]{x} \cdot \ln(7x))$$
$$f(x) = 5 \cdot \sin x + \sin(x+5) + \sin(5x) + \sin^5 x + 5^{\sin x}$$
$$f(x) = \sin 5 + x^{\sin 5} + x \cdot \sin 5$$

Solution. • $f(x) = (4x-2)^6 + \sqrt[3]{x - x^7}$

$$f'(x) = \left((4x-2)^6\right)' + \left((x - x^7)^{\frac{1}{3}}\right)' = (\star)$$

$$f_1(x) = (4x-2)^6, \quad k(x) = x^6, \quad b(x) = 4x - 2$$

$$\Rightarrow k \circ b(x) = k(b(x)) = k(4x-2) = f_1(x)$$

$$f_1'(x) = k'(b(x)) \cdot b'(x) = 6 \cdot (4x-2)^5 \cdot 4 = 24 \cdot (4x-2)^5$$

In a similar way with the notation $f_2(x) = (x-x^7)^{\frac{1}{3}}$ we get that

$$f_2'(x) = \frac{1}{3} \cdot (x-x^7)^{-\frac{2}{3}} \cdot (1-7x^6) = \frac{1}{3} \cdot \frac{1-7x^6}{\sqrt[3]{(x-x^7)^2}}$$

$$(\star) = f_1'(x) + f_2'(x) = 24 \cdot (4x-2)^5 + \frac{1}{3} \cdot \frac{1-7x^6}{\sqrt[3]{(x-x^7)^2}}$$

- $f(x) = \operatorname{tg}(-x) + \operatorname{arctg}(1-x^2)$

$$f'(x) = (\operatorname{tg}(-x))' + (\operatorname{arctg}(1-x^2))' =$$

$$= \frac{1}{\cos^2(-x)} \cdot (-1) + \frac{1}{1+(1-x^2)^2} \cdot (-2x) = -\frac{1}{\cos^2(-x)} - \frac{2x}{1+(1-x^2)^2}$$

- $f(x) = e^x + e^{-x} - e^{-x} + e^{3x} - e^{-2}$

$$f'(x) = e^x - e^{-x} + e^{-x} + 3 \cdot e^{3x}$$

- $f(x) = 6^{4x-2} + \frac{\sin(13x^2)}{\operatorname{tg}\left(\frac{x}{5}\right)}$

$$f'(x) = (6^{4x-2})' + \left(\frac{\sin(13x^2)}{\operatorname{tg}\left(\frac{x}{5}\right)}\right)' =$$

$$= 6^{4x-2} \cdot \ln 6 \cdot (4x-2)' + \frac{(\sin(13x^2))' \cdot \operatorname{tg}\left(\frac{x}{5}\right) - \sin(13x^2) \cdot \left(\operatorname{tg}\left(\frac{x}{5}\right)\right)'}{\left(\operatorname{tg}\left(\frac{x}{5}\right)\right)^2} =$$

$$= 6^{4x-2} \cdot \ln 6 \cdot 4 + \frac{\cos(13x^2) \cdot 26x \cdot \operatorname{tg}\left(\frac{x}{5}\right) - \sin(13x^2) \cdot \frac{1}{\cos^2\left(\frac{x}{5}\right)} \cdot \frac{1}{5}}{\left(\operatorname{tg}\left(\frac{x}{5}\right)\right)^2}$$

- $f(x) = \cos^3(\sqrt[10]{x} \cdot \ln(7x))$

$$f'(x) = \left(\left(\cos(\sqrt[10]{x} \cdot \ln(7x))\right)^3\right)' =$$

$$= 3 \cdot \left(\cos(\sqrt[10]{x} \cdot \ln(7x))\right)^2 \cdot \left(\cos(\sqrt[10]{x} \cdot \ln(7x))\right)' =$$

$$= 3 \cdot \left(\cos(\sqrt[10]{x} \cdot \ln(7x))\right)^2 \cdot \left(-sin(\sqrt[10]{x} \cdot \ln(7x))\right) \cdot (\sqrt[10]{x} \cdot \ln(7x))' =$$

$$= 3 \cdot \left(\cos(\sqrt[10]{x} \cdot \ln(7x))\right)^2 \cdot \left(-sin(\sqrt[10]{x} \cdot \ln(7x))\right) \cdot \left(\frac{1}{10} \cdot \frac{1}{\sqrt[10]{x^9}} \cdot \ln(7x) + \sqrt[10]{x} \cdot \frac{1}{7x} \cdot 7\right)$$

- $f(x) = 5 \cdot \sin x + \sin(x+5) + \sin(5x) + \sin^5 x + 5^{\sin x}$

$$f'(x) = 5 \cdot \cos x + \cos(x+5) \cdot 1 + 5 \cdot \cos(5x) + 5 \sin^4 x \cdot \cos x + 5^{\sin x} \cdot \ln 5 \cdot \cos x$$

- $f(x) = \sin 5 + x^{\sin 5} + x \cdot \sin 5$

$$f'(x) = 0 + \sin 5 \cdot x^{(\sin 5)-1} + \sin 5 = \sin 5 \cdot x^{(\sin 5)-1} + \sin 5$$

Exercise 1.1.3 *Differentiate the following functions:*

$$H(t) = t^3(t^2-1)(t-1)^2$$
$$K(q) = (q \cdot \sin q)^9$$

Solution. • $H(t) = t^3(t^2-1)(t-1)^2$

$$H'(t) = \left[t^3(t^2-1)\right]' \cdot (t-1)^2 + t^3(t^2-1) \cdot \left[(t-1)^2\right]' =$$

$$= \left(3t^2(t^2-1) + t^3 \cdot 2t\right) \cdot (t-1)^2 + t^3(t^2-1) \cdot 2 \cdot (t-1)$$

- $K(q) = (q \cdot \sin q)^9$

$$K'(q) = 9 \cdot (q \cdot \sin q)^8 \cdot (q \cdot \sin q)' =$$

$$= 9 \cdot (q \cdot \sin q)^8 \cdot (1 \cdot \sin q + q \cdot \cos q)$$

Exercise 1.1.4 *Differentiate the following functions and give the value $f'(x_0)$.*

$$f(x) = \ln(\sin x) \quad x_0 = \frac{\pi}{2}$$
$$f(x) = e^{x^2} \quad x_0 = 3$$

Solution. • $f(x) = \ln(\sin x) \quad x_0 = \frac{\pi}{2}$

$$f'(x) = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \text{ctg } x \quad \Rightarrow f'(x_0) = f'\left(\frac{\pi}{2}\right) = \text{ctg}\frac{\pi}{2} = 0$$

• $f(x) = e^{x^2} \quad x_0 = 3$

$$f'(x) = e^{x^2} \cdot 2x \quad \Rightarrow f'(x_0) = f'(3) = e^{3^2} \cdot 2 \cdot 3 = 6 \cdot e^9$$

Exercise 1.1.5 *Give the second order derivative function:*

$$f(x) = \text{arctg } x$$
$$f(x) = x \cdot \ln x$$

Solution. • $f(x) = \text{arctg } x$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = \left(\frac{1}{1+x^2}\right)' = \left((1+x^2)^{-1}\right)' = -(1+x^2)^{-2} \cdot 2x = -\frac{2x}{(1+x^2)^2}$$

• $f(x) = x \cdot \ln x$

$$f'(x) = (x)' \cdot \ln x + x \cdot (\ln x)' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$f''(x) = (\ln x + 1)' = \frac{1}{x} \quad (x > 0)$$

Exercise 1.1.6 *Give the third order derivative function:*

$$f(x) = \cos(8x)$$
$$f(x) = \sqrt{x}$$

• $f(x) = \cos(8x)$

$$f'(x) = -8\sin(8x) \;\Rightarrow f''(x) = -8^2\cos(8x) \Rightarrow f'''(x) = 8^3\sin(8x)$$

- $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow f''(x) = \left(\frac{1}{2}x^{-\frac{1}{2}}\right)' = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4}\frac{1}{\sqrt{x^3}}$$

$$\Rightarrow f'''(x) = \left(-\frac{1}{4}x^{-\frac{3}{2}}\right)' = \frac{3}{8}x^{-\frac{5}{2}} = \frac{3}{8}\frac{1}{\sqrt{x^5}}$$

Exercise 1.1.7 *Let $f(x) = \frac{2x+1}{\sqrt{x^2+8}}$ be given. Give the equation of the tangent line at $x_0 = -1$.*

The equation of the tangent line at x_0:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x_0) = f(-1) = \frac{-2+1}{\sqrt{(-1)^2+8}} = -\frac{1}{3}$$

$$f'(x) = \left(\frac{2x+1}{\sqrt{x^2+8}}\right)' = \frac{2\cdot(\sqrt{x^2+8}) - (2x+1)\cdot\frac{1}{2}(x^2+8)^{-\frac{1}{2}}\cdot 2x}{x^2+8} =$$

$$= \frac{2\cdot(\sqrt{x^2+8}) - (2x+1)\cdot\frac{x}{\sqrt{x^2+8}}}{x^2+8}$$

$$f'(x_0) = f'(-1) = \frac{2\cdot 3 - (-1)\cdot\frac{(-1)}{3}}{9} = \frac{6-\frac{1}{3}}{9} = \frac{17}{3\cdot 9} = \frac{17}{27}$$

The equation of the tangent line at $x_0 = -1$:

$$y = -\frac{1}{3} + \frac{17}{27}\cdot(x+1)$$

Exercise 1.1.8 *Let $f(x) = arctg\ x$ be given. Give the equation of the tangent line at $x_0 = 1$. Give the value arctg $0,9$ approximately by linear approximation.*

$$f(x) = \text{arctg } x \quad f(x_0) = \frac{\pi}{4} \approx 0.79$$

$$f'(x) = \frac{1}{1+x^2} \quad f'(x_0) = 0.5$$

$$f''(x) = -\frac{2x}{(1+x^2)^2} \quad f''(x_0) = -0.5$$

- The equation of the tangent line: $y = 0.5(x-1) + 0.79$.
- arctg $0.9 \approx 0.5(0.9-1) + 0.79 = 0.74$.

1.2 Analysis of differentiable functions

Steps of investigations of differentiable functions:

1. If the domain is not given, to give the largest subsets of the reals on which the function (the expression) can be defined.
2. To give the places where the function is not continuous.
3. To give the zero set of the function.
4. To give the limits, where the function is not continuous. To give the limits on the border of the domain.
5. To investigate the monotonicity, and to give the places and values of the local extremums.(by the first derivative function)
6. To investigate the convexity and inflexion points. (by the second derivative function)
7. To give the schematic graph of the function using the last steps.
8. To give the range of the function.

Exercise 1.2.1 *Give the analysis of the following functions:*

$$f(x) = x^3 - 3x$$

1. The largest subsets of the reals on which the function (the expression) can be defined: $x \in \mathbb{R}$
2. The places where the function is continuous:$x \in \mathbb{R}$.
3. The zero set of the function: $\{-\sqrt{3}, 0, \sqrt{3}\}$
   $f(x) = 0 \Leftrightarrow x^3 - 3x = 0 \Leftrightarrow x \cdot (x^2 - 3) = 0.$. Thus $x = 0$ or $x = \pm\sqrt{3}$.
4. $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} (x^3 - 3x) = \lim\limits_{x \to -\infty} x^3 \cdot \left(1 - \frac{3}{x^2}\right) = -\infty$
   $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} (x^3 - 3x) = \lim\limits_{x \to \infty} x^3 \cdot \left(1 - \frac{1}{x^2}\right) = \infty.$
5. $f'(x) = 3x^2 - 3$

Possible places for extremum: We have to find the places where $f' = 0$.

$$3x^2 - 3 = 0 \quad \Rightarrow \quad 3x^2 = 3 \quad \Rightarrow \quad x^2 = 1,$$

So it follows that $x = \pm 1$.

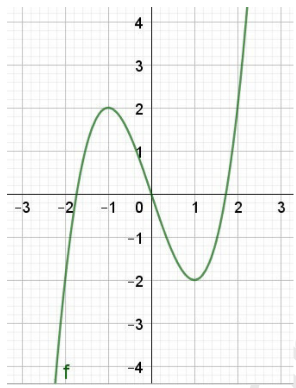
	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\uparrow$	loc. max.	$\downarrow$	loc. min.	$\uparrow$

6.

Possible places for inflexion points: $f'' = 0 \;\Leftrightarrow\; 6x = 0 \;\Leftrightarrow\; x = 0.$

	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	$-$	0	$+$
$f(x)$	concave	ip	convex

7. The schematic graph of the function:



8. The range of the function: $f(x) \in \mathbb{R}$.

Exercise 1.2.2 *Give the analysis of the following functions:*

$$f(x) = x + \frac{1}{x}$$

1. The largest subsets of the reals on which the function (the expression) can be defined: $x \in \mathbb{R} \setminus \{0\}$

2. The places where the function is not continuous:$x = 0$.

3. The zero set of the function is empty. $(f(x) = 0 \Leftrightarrow x + \frac{1}{x} = 0 \Leftrightarrow x^2 + 1 = 0)$.

4.
$$\lim_{x \to -\infty} f(x) = -\infty, \quad \lim_{x \to +\infty} f(x) = +\infty,$$
$$\lim_{x \to 0-0} f(x) = -\infty, \quad \lim_{x \to 0+0} f(x) = +\infty$$

5. $f'(x) = (x + x^{-1})' = 1 - x^{-2} = 1 - \frac{1}{x^2}$

Possible places for extremum: We have to find the places where $f' = 0$.

$$\underline{f'(x) = 0} \Leftrightarrow 1 - \frac{1}{x^2} = 0 \ \Leftrightarrow \frac{x^2-1}{x^2} = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow \ \underline{x = -1 \text{ or } x = 1}.$$

$$\underline{f'(x) > 0} \Leftrightarrow 1 - \frac{1}{x^2} > 0 \ \Leftrightarrow \frac{x^2-1}{x^2} > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow \ \underline{x < -1 \text{ or } x > 1}.$$

$$\underline{f'(x) < 0} \Leftrightarrow 1 - \frac{1}{x^2} < 0 \ \Leftrightarrow \frac{x^2-1}{x^2} < 0 \Leftrightarrow x^2 - 1 < 0 \Leftrightarrow \ \underline{-1 < x < 1.}$$

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$f'(x)$	$+$	0	$-$	$\nexists$	$-$	0	$+$
$f(x)$	$\uparrow$	loc. max.	$\downarrow$	$\nexists$	$\downarrow$	loc. min.	$\uparrow$

The value of the local maximum: $f(-1) = -2$, the value of the local minimum: $f(1) = 2$

6. $f''(x) = (1 - x^{-2})' = -(-2)x^{-3} = \frac{2}{x^3}$.

Possible places for inflexion points: $f'' = 0$.

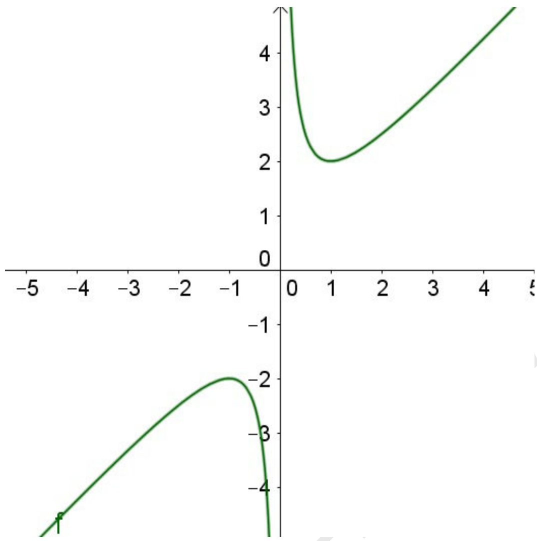
$$f''(x) \neq 0 : \text{ f has no inflection point}$$

$\underline{f''(x) > 0} \Leftrightarrow \frac{2}{x^3} > 0 \Leftrightarrow \underline{x > 0}$, on similar way $\underline{f''(x) < 0} \Leftrightarrow \frac{2}{x^3} < 0 \Leftrightarrow \underline{x < 0}$
This means that f is convex on the interval $(0, +\infty)$ and concave on $(-\infty, 0)$.

7. The schematic graph of the function:

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$f'(x)$	$+$	0	$-$	$\nexists$	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	$\nexists$	$+$	$+$	$+$
$f(x)$	$\uparrow\cap$	loc. max.	$\downarrow\cap$	$\nexists$	$\downarrow\cup$	loc. min.	$\uparrow\cup$

8. The range of the function: $f(x) \in (-\infty, -2] \cup [2, +\infty)$



Exercise 1.2.3 *Determine the global maximum and minimum of the following functions.*

Solution:

- $f(x) = \frac{x}{x^2-4}, \quad x \in [-1, 1]$

$D = [-1, 1]$ is a bounded set, therefore the continuous function f have global maximum and minimum; these are either in the interior point of the set D ($D^o =]-1, 1[$) as local extremums or on the border of the set D.

I. **On the set** D^o

$$f'(x) = \frac{1 \cdot (x^2 - 4) - x \cdot 2x}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2}$$

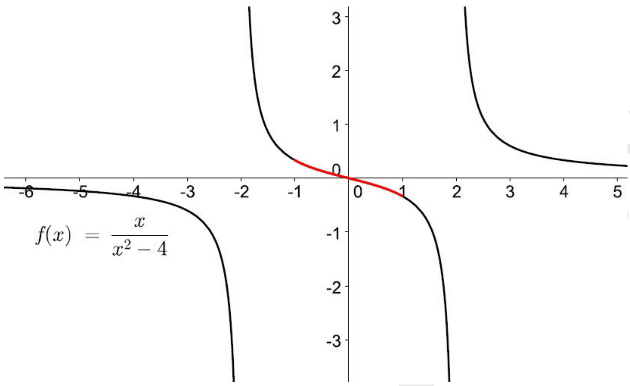
$$f'(x) = 0 \Leftrightarrow -x^2 - 4 = 0 \Leftrightarrow x^2 = -4$$

$\Rightarrow$ Thus f has no local extremum on D^o (so there are no global extremum on D^o).

II. **On the border of D**

$$f(-1) = \frac{-1}{(-1)^2 - 4} = \frac{1}{3} \text{ and } f(1) = \frac{1}{1^2 - 4} = -\frac{1}{3}$$

Thus the maximum of f is $\frac{1}{3}$, the minimum of f is $-\frac{1}{3}$. • $f(x) =$



e^{2x-x^2}, $x \in [0, 4]$
$D = [0, 4]$ is a bounded set, therefore the continuous function f have global maximum and minimum; these are either in the interior point of the set D ($D^o =]0, 4[$) as local extremums or on the border of the set D.

I. **On the set D^o**

$$f'(x) = e^{2x-x^2} \cdot (2 - 2x)$$

$$f'(x) = 0 \Leftrightarrow 2 - 2x = 0 \Leftrightarrow x = 1 \in D^o$$

$$f'(x) > 0 \Leftrightarrow 2 - 2x > 0 \Leftrightarrow x < 1$$

$$f'(x) < 0 \Leftrightarrow 2 - 2x < 0 \Leftrightarrow x > 1$$

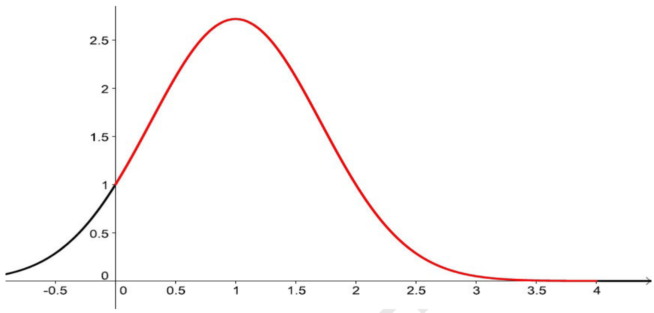
	$x < 1$	$x = 1$	$x > 1$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	loc.max.	$\downarrow$

$$f(1) = e^{2 \cdot 1 - 1^2} = e^1 = e$$

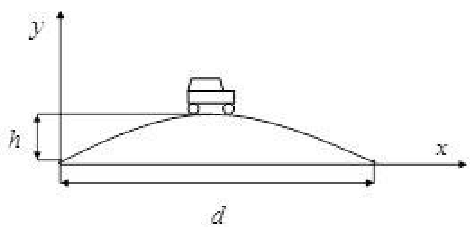
II. **On the border of D**

$$f(0) = e^0 = 1 \text{ and } f(4) = e^{-8} = \frac{1}{e^8}$$

Thus local maximum is e which is the global maximum of f, and $\frac{1}{e^8}$ is the global minimum of f.



Exercise 1.2.4 *The shape of a bridge over a river can be written by the equation $y = -\frac{h}{1600}x^2 + \frac{h}{20}x$, where h denotes the height of the bridge. What is the maximal height of the bridge if the maximal lead angle of the bridge is $30°$ and the width of the river is $d = 80$ $[m]$?*



Solution.

Denote m_P the slope of the tangent line at $P = (x; y)$ determined by the parabola $y = -\frac{h}{1600}x^2 + \frac{h}{20}x$. Then we have the assumption

$$|m_P| \le tg\ 30°.$$

The bridge is the image of the function

$$f(x) = -\frac{h}{1600}x^2 + \frac{h}{20}x, \quad x \in [0;80]$$

, and $f'(x)$ is the slope of the tangent line at $P = (x, y)$. As the polynomial function

$$f'(x) = -\frac{h}{800}x + \frac{h}{20}, \quad x \in [0;80]$$

is continuous on the bounded closed interval $[0;80]$, thus it has a global maximum and minimum. Because the function f' is strictly monotone, then the global maximum an minimum are in the endpoints of the interval:

$$f'(0) = \frac{h}{20} \text{ the maximum of the slopes: } max_{m_P},$$

$$f'(80) = -\frac{h}{10} + \frac{h}{20} = -\frac{2h}{20} + \frac{h}{20} = -\frac{h}{20} \text{ the minimum of the slopes:.}$$

Therefore $\frac{h}{20} \leq tg\ 30° = \frac{\sqrt{3}}{3}$ must be satisfied, which from

$$h \leq 20\frac{\sqrt{3}}{3} \approx 11,547\ [m].$$

1.3 L'Hospital's rule, Taylor polynomials.

L'Hospital's rule states that if $f, g\colon (x_0, x_0 + r) \to \mathbb{R}$ are differentiable functions, such that $g(x) \cdot g'(x) \neq 0$ and either $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$, or $\lim\limits_{x\to x_0} f(x) = \pm\infty$, $\lim\limits_{x\to x_0} g(x) = \pm\infty$ (possibly $x_0 = \pm\infty$), then we have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)},$$

if the latter limit exists.

Let $f:\ [a, b] \to \mathbb{R}$ be a function differentiable n times. Then the Taylor polynomial of order n of $f(x)$ around a point $x_0 \in (a, b)$ is given by

$$T_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

If $x_0 = 0$ then the above polynomial is called the MacLaurin polynomial of $f(x)$ of order n. ($0!\!:=\!1$ and $f^{(0)} := f$)

Exercise 1.3.1 *Calculate the following limits.*

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} \qquad \lim_{x\to 1}\frac{x^3-1}{4x^3-x-3} \qquad \lim_{x\to+\infty}\frac{x^3-1}{4x^3-x-3}$$

$$\lim_{x\to\frac{\pi}{4}}\frac{sin\ x - cos\ x}{x-\frac{\pi}{4}} \qquad \lim_{x\to 0}\frac{sin\ x}{x} \qquad \lim_{x\to+\infty} x\cdot sin\frac{1}{x}$$

$$\lim_{x\to 0}\frac{\sin 4x}{5x} \qquad \lim_{x\to 0}\frac{1-\cos 2x}{1-\cos 3x} \qquad \lim_{x\to 2}\frac{\ln\left(\frac{x}{2}\right)}{x-2}$$

$$\lim_{x\to\infty} x\mathrm{e}^{-x}$$

Solution.

- $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{(\sqrt{1+x}-1)'}{(x)'} = \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}\cdot 1}{1} =$$

  $$= \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{\frac{1}{2\sqrt{1+0}}}{1} = \frac{1}{2}$$

- $$\lim_{x\to 1}\frac{x^3-1}{4x^3-x-3} =$$

  $$= \lim_{x\to 1}\frac{(x^3-1)'}{(4x^3-x-3)'} = \lim_{x\to 1}\frac{3x^2}{12x^2-1} = \frac{3\cdot 1^2}{12\cdot 1^2-1} = \frac{3}{11}$$

- $$\lim_{x\to+\infty}\frac{x^3-1}{4x^3-x-3} =$$

  $$= \lim_{x\to+\infty}\frac{(x^3-1)'}{(4x^3-x-3)'} = \lim_{x\to+\infty}\frac{3x^2}{12x^2-1} =$$

  $$= \lim_{x\to+\infty}\frac{(3x^2)'}{(12x^2-1)'} = \lim_{x\to+\infty}\frac{6x}{24x} =$$

  $$= \lim_{x\to+\infty}\frac{(6x)'}{(24x)'} = \lim_{x\to+\infty}\frac{6}{24} = \frac{6}{24} = \frac{1}{4}$$

- $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{sin\ x - cos\ x}{x - \frac{\pi}{4}} =$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(sin\ x - cos\ x)'}{(x - \frac{\pi}{4})'} = \lim_{x \to \frac{\pi}{4}} \frac{cos\ x + sin\ x}{1} = \lim_{x \to \frac{\pi}{4}} \frac{cos\ \frac{\pi}{4} + sin\ \frac{\pi}{4}}{1} = \sqrt{2}$$

- $\lim\limits_{x \to 0} \dfrac{sin\ x}{x} = \lim\limits_{x \to 0} \dfrac{(sin\ x)'}{(x)'} = \lim\limits_{x \to 0} \dfrac{cos\ x}{1} = \dfrac{cos\ 0}{1} = 1$

- $\lim\limits_{x \to +\infty} x \cdot sin \dfrac{1}{x} =$

$$= \lim_{x \to +\infty} \frac{sin \frac{1}{x}}{\frac{1}{x}} = \lim_{x \to +\infty} \frac{(sin \frac{1}{x})'}{(\frac{1}{x})'} = \lim_{x \to +\infty} \frac{cos \frac{1}{x} \cdot (-\dfrac{1}{x^2})}{-\frac{1}{x^2}} =$$

$$= \lim_{x \to +\infty} \cos \frac{1}{x} = 1$$

- $\lim\limits_{x \to 0} \dfrac{\sin 4x}{5x} = \lim\limits_{x \to 0} \dfrac{(\sin 4x)'}{(5x)'} = \lim\limits_{x \to 0} \dfrac{4 \cos 4x}{5} = \frac{4}{5}$
- $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{1 - \cos 3x} = \lim\limits_{x \to 0} \dfrac{(1 - \cos 2x)'}{(1 - \cos 3x)'} = \lim\limits_{x \to 0} \dfrac{2 \sin 2x}{3 \sin 3x} =$

$$= \lim_{x \to 0} \frac{(2 \sin 2x)'}{(3 \sin 3x)'} = \lim_{x \to 0} \frac{4 \cos 2x}{9 \cos 3x} = \frac{4}{9}$$

- $\lim\limits_{x \to 2} \dfrac{\ln\left(\frac{x}{2}\right)}{x - 2} = \lim\limits_{x \to 2} \frac{1}{x} = \frac{1}{2}$
- $\lim\limits_{x \to \infty} x e^{-x} = \lim\limits_{x \to \infty} \dfrac{x}{e^x} = \lim\limits_{x \to \infty} \dfrac{1}{e^x} = 0$

Exercise 1.3.2 *Give the Taylor polynomial of $f(x) = arctg\ x$ of order* 3 *around the point $x_0 = 1$. By the help of the result give the value arctg* $0,9$ *approximately.*

Solution.

$$T_{n,x_0}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$T_{3,1}(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$f(x) = \operatorname{arctg} x \;\Rightarrow\; f(x_0) = \frac{\pi}{4} \approx 0.79$$

$$f'(x) = \frac{1}{1+x^2} \;\Rightarrow\; f'(x_0) = 0.5$$

$$f''(x) = -\frac{2x}{(1+x^2)^2} \;\Rightarrow\; f''(x_0) = -0.5$$

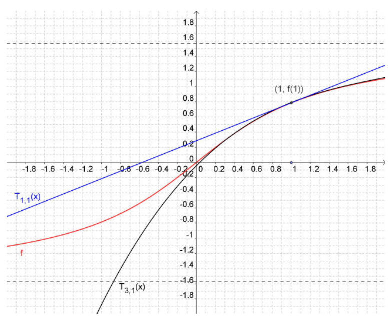
$$f'''(x) = \frac{6x^2-2}{(1+x^2)^3} \;\Rightarrow\; f'''(x_0) = 0.5$$

$$T_{3,1}(x) = 0.79 + 0.5(x-1) - 0.25(x-1)^2 + \frac{1}{12}(x-1)^3$$

$$\operatorname{arctg} 0.9 \approx T_{3,1}(0.9) \approx 0.79 + 0.5(0.9-1) - 0.25(0.9-1)^2 + \frac{1}{12}(0.9-1)^3 =$$

$$= 0.79 + 0.5(-0.1) - 0.25(-0.1)^2 + \frac{1}{12}(-0.1)^3 \approx$$

$$\approx 0.79 - 0.05 - 0.0025 - 0.000083 = 0.7374$$



Exercise 1.3.3 *Give the polynomial $f(x) = x^3 + 5x^2 - 7x + 3$ as the polynomial of $x - 1$.*

Solution. Now $x_0 = 1$.

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$f'(x) = 3x^2 + 10x - 7,\ f''(x) = 6x + 10,\ f'''(x) = 6$

$$f(1) = 2, \quad f'(1) = 6, \quad f''(1) = 16, \quad f'''(1) = 6$$

$f(x) = (x-1)^3 + 8(x-1)^2 + 6(x-1) + 2$

2

Integration of univariable functions

2.1 Indefinit integral

$f(x)$	k	x^α	$\cos x$	$\sin x$	$\frac{1}{\cos^2 x}$	$\frac{1}{\sin^2 x}$
$\int f(x)\,\mathrm{d}x$	$kx+c$	$\frac{x^{\alpha+1}}{\alpha+1}+c$	$\sin x + c$	$-\cos x + c$	$\tan x + c$	$-\cot x + c$

$f(x)$	$\frac{1}{1+x^2}$	$\frac{1}{\sqrt{1-x^2}}$	e^x	a^x	$\frac{1}{x}$	$\cosh x$	$\sinh x$
$\int f(x)\,\mathrm{d}x$	$\arctan(x)$	arcsin(x)+c	e^x	$\frac{a^x}{\ln a}$	$\ln\lvert x\rvert$	$\sinh x$	$\cosh x$

Exercise 2.1.1 *Determine the following indefinite integrals:*

$$\int \left(\frac{4}{\sin^2 x} + 12^x - 3,4\cdot x^{12}\right)\mathrm{d}x \qquad \int \left(\frac{4,6}{\sqrt[3]{x}} + \frac{9}{\sqrt{1-x^2}} - \frac{10}{x^2+1}\right)\mathrm{d}x$$

Solution • $\int \left(\frac{4}{\sin^2 x} + 12^x - 3,4\cdot x^{12}\right)\mathrm{d}x =$

$$= 4\cdot\int \frac{1}{\sin^2 x}\,\mathrm{d}x + \int 12^x\,\mathrm{d}x - 3,4\cdot\int x^{12}\,\mathrm{d}x =$$

$$= 4\cdot(-\mathrm{ctg}x) + c_1 + \frac{12^x}{\ln 12} + c_2 - 3,4\cdot\frac{x^{13}}{13} + c_3 =$$

$$= -4 \cdot \mathrm{ctg} x + \frac{12^x}{\ln 12} - \frac{34 \cdot x^{13}}{130} + c$$

where c is an arbitrary real number.

- $\int \left(\frac{4,6}{\sqrt[3]{x}} + \frac{9}{\sqrt{1-x^2}} - \frac{10}{x^2+1} \right) \mathrm{d}x =$

$$= 4,6 \cdot \int x^{-\frac{1}{3}} \,\mathrm{d}x + 9 \cdot \int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x - 10 \cdot \int \frac{1}{x^2+1} \,\mathrm{d}x =$$

$$= 4,6 \cdot \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + 9 \cdot \arcsin x - 10 \cdot \mathrm{arctg} x + c =$$

$$= \frac{46}{10} \cdot \frac{3}{2} \cdot \sqrt[3]{x^2} + 9 \cdot \arcsin x - 10 \cdot \mathrm{arctg} x + c =$$

$$= \frac{69}{10} \cdot \sqrt[3]{x^2} + 9 \cdot \arcsin x - 10 \cdot \mathrm{arctg} x + c =$$

$$= 6,9 \cdot \sqrt[3]{x^2} + 9 \cdot \arcsin x - 10 \cdot \mathrm{arctg} x + c$$

where c is an arbitrary real number.

Exercise 2.1.2 *Determine $\int f$ with the help of the rule*

$$\int f(ax+b) \,\mathrm{d}x = \frac{F(ax+b)}{a} + c, \text{ where } F' = f$$

$$f(x) = \cos(2x+3) \qquad f(x) = \sin(\pi x - e) \qquad f(x) = \frac{1}{\cos^2(5x-2)}$$

$$f(x) = \frac{1}{\sin^2(3x+6)} \qquad f(x) = \mathrm{e}^{4x} \qquad f(x) = \frac{1}{10-5x}$$

Solution.

- $\int \cos(2x+3) \,\mathrm{d}x = \frac{\sin(2x+3)}{2} + c$, $(f(x) = \cos(x))$

- $\int \sin(\pi x - e) \,\mathrm{d}x = -\frac{\cos(\pi x - e)}{\pi} + c$, $(f(x) = \sin(x))$

- $\displaystyle\int \frac{1}{\cos^2(5x-2)}\,\mathrm{d}x = \frac{\mathrm{tg}\,(5x-2)}{5} + c,\ (f(x) = \tfrac{1}{\cos^2(x)})$

- $\displaystyle\int \frac{1}{\sin^2(3x+6)}\,\mathrm{d}x = -\frac{\mathrm{ctg}\,(3x+6)}{3} + c,\ (f(x) = \tfrac{1}{\sin^2(x)})$

- $\displaystyle\int \mathrm{e}^{4x}\,\mathrm{d}x = \frac{\mathrm{e}^{4x}}{4} + c,\ (f(x) = e^x)$

- $\displaystyle\int \frac{1}{10-5x}\,\mathrm{d}x = \frac{\ln|10-5x|}{-5} + c,\ (f(x) = \tfrac{1}{x})$

Exercise 2.1.3 *Determine the indefinite integrals* $\int f$ *by*

$$\int g' \cdot g^n\,\mathrm{d}x = \frac{g^{n+1}}{n+1} + c, \quad (n \neq -1))$$

$$f(x) = \sin^3 x \cdot \cos x \qquad f(x) = \cos^4(6x)\cdot\sin(6x)$$

$$\frac{\sin(-x)}{\sqrt[10]{\cos^7(-x)}} \qquad f(x) = \frac{\ln^5 x}{x}$$

Solution.

- $\displaystyle\int \sin^3 x \cdot \cos x\,\mathrm{d}x = \frac{\sin^4 x}{4} + c$
- $g(x) = \cos(6x), \quad n = 4, \quad g'(x) = -6\cdot\sin(6x)$

$$\int \cos^4(6x)\cdot\sin(6x)\,\mathrm{d}x = -\frac{1}{6}\int \cos^4(6x)\cdot(-6)\cdot\sin(6x)\,\mathrm{d}x = -\frac{1}{6}\cdot\frac{\cos^5(6x)}{5} + c$$

- $g(x) = \cos(-x), \quad n = -\frac{7}{10}, \quad g'(x) = -\sin(-x)\cdot(-1) = \sin(-x)$

$$\int \frac{\sin(-x)}{\sqrt[10]{\cos^7(-x)}}\,\mathrm{d}x = \int \sin(-x)\cdot(\cos(-x))^{-\frac{7}{10}}\,\mathrm{d}x = \frac{10}{3}\cdot(\cos(-x))^{\frac{3}{10}} + c =$$

$$= \frac{10}{3}\cdot\ \sqrt[10]{\cos^3(-x)} + c$$

- $\displaystyle\int \frac{\ln^5 x}{x}\,\mathrm{d}x = \int \frac{1}{x}\cdot(\ln x)^5\,\mathrm{d}x = \frac{\ln^6 x}{6} + c$

Exercise 2.1.4 *Determine the indefinite integrals $\int f$ by*

$$\int \frac{g'}{g}\,\mathrm{d}x = \ln|g| + c$$

- $\displaystyle\int \frac{x}{x^2-5}\,\mathrm{d}x = \frac{1}{2}\cdot\int \frac{2\cdot x}{x^2-5}\,\mathrm{d}x = \frac{1}{2}\cdot\ln|x^2-5| + c$
- $\displaystyle\int \frac{x+2}{x^2+4x+1}\,\mathrm{d}x = \frac{1}{2}\cdot\int \frac{2\cdot(x+2)}{x^2+4x+1}\,\mathrm{d}x = \frac{1}{2}\cdot\ln|x^2+4x+1| + c$
- $\displaystyle\int \frac{e^{9x}}{8\cdot e^{9x}+9}\,\mathrm{d}x = \frac{1}{72}\cdot\int \frac{72\cdot e^{9x}}{8\cdot e^{9x}+9}\,\mathrm{d}x = \frac{1}{72}\cdot\ln|8\cdot e^{9x}+9| + c$
- $\displaystyle\int \mathrm{tg}(-3x)\,\mathrm{d}x = \int \frac{\sin(-3x)}{\cos(-3x)}\,\mathrm{d}x =$
$$= \frac{1}{3}\cdot\int \frac{3\cdot\sin(-3x)}{\cos(-3x)}\,\mathrm{d}x = \frac{1}{3}\cdot\ln|\cos(-3x)| + c$$
- $\displaystyle\int \frac{1}{\mathrm{tg}x\cdot\cos^2 x}\,\mathrm{d}x = \int \frac{\frac{1}{\cos^2 x}}{\mathrm{tg}x}\,\mathrm{d}x = \ln|\mathrm{tg}x| + c$

Exercise 2.1.5 *Determine the following indefinite integrals. (partial integration)*

$$\int (4x-7)\cdot 3^x\,\mathrm{d}x = ? \qquad \int (x+5)\cdot\cos(4-11x)\,\mathrm{d}x = ?$$

$$\int \frac{x}{\mathrm{e}^x}\,\mathrm{d}x = ? \qquad \int (x^2-3x+1)\cdot\sin(5x)\,\mathrm{d}x = ?$$

Solution:

$$\int (4x-7)\cdot 3^x\,\mathrm{d}x = \left[\begin{array}{rr} g(x) = 4x-7 & f'(x) = 3^x \\ g'(x) = 4 & f(x) = \frac{3^x}{\ln 3} \end{array}\right] =$$

$$= \frac{3^x}{\ln 3}\cdot(4x-7) - \int \frac{3^x}{\ln 3}\cdot 4\,\mathrm{d}x = \frac{3^x}{\ln 3}\cdot(4x-7) - \frac{4}{\ln 3}\cdot\int 3^x\,\mathrm{d}x =$$
$$= \frac{3^x}{\ln 3}\cdot(4x-7) - \frac{4}{\ln 3}\cdot\frac{3^x}{\ln 3} + c$$

$$\int (x+5)\cdot\cos(4-11x)\,\mathrm{d}x = \left[\begin{array}{rr} g(x)=x+5 & f'(x)=\cos(4-11x) \\ g'(x)=1 & f(x)=\frac{\sin(4-11x)}{-11} \end{array}\right] =$$

$$= \frac{\sin(4-11x)}{-11}\cdot(x+5) - \int \frac{\sin(4-11x)}{-11}\cdot 1\,\mathrm{d}x =$$

$$= \frac{\sin(4-11x)}{-11}\cdot(x+5) + \frac{1}{11}\cdot\int \sin(4-11x)\,\mathrm{d}x =$$

$$= \frac{\sin(4-11x)}{-11}\cdot(x+5) + \frac{1}{11}\cdot\frac{-\cos(4-11x)}{-11} + c$$

$$\int \frac{x}{\mathrm{e}^x}\,\mathrm{d}x = \int x\cdot\mathrm{e}^{-x}\,\mathrm{d}x = \left[\begin{array}{rr} g(x)=x & f'(x)=\mathrm{e}^{-x} \\ g'(x)=1 & f(x)=\frac{\mathrm{e}^{-x}}{-1}=-\mathrm{e}^{-x} \end{array}\right] =$$

$$= -\mathrm{e}^{-x}\cdot x - \int -\mathrm{e}^{-x}\cdot 1\,\mathrm{d}x =$$

$$= -\mathrm{e}^{-x}\cdot x + \int \mathrm{e}^{-x}\,\mathrm{d}x = -\mathrm{e}^{-x}\cdot x - \mathrm{e}^{-x} + c =$$

$$= -\frac{x}{e^x} - \frac{1}{e^x} + c$$

$$\int (x^2-3x+1)\cdot\sin(5x)\,\mathrm{d}x = \left[\begin{array}{rr} g(x)=x^2-3x+1 & f'(x)=\sin(5x) \\ g'(x)=2x-3 & f(x)=\frac{-\cos(5x)}{5} \end{array}\right] =$$

$$= \frac{-\cos(5x)}{5}\cdot(x^2-3x+1) + \frac{1}{5}\int \cos(5x)\cdot(2x-3)\,\mathrm{d}x =$$

$$= \left[\begin{array}{rr} g(x)=2x-3 & f'(x)=\cos(5x) \\ g'(x)=2 & f(x)=\frac{\sin(5x)}{5} \end{array}\right] =$$

$$= \frac{-\cos(5x)}{5}\cdot(x^2-3x+1) + \frac{1}{5}\cdot\left(\frac{\sin(5x)}{5}\cdot(2x-3) - \int \frac{\sin(5x)}{5}\cdot 2\,\mathrm{d}x\right) =$$

$$= \frac{-\cos(5x)}{5}\cdot(x^2-3x+1) + \frac{1}{5}\cdot\left(\frac{\sin(5x)}{5}\cdot(2x-3) - \frac{2}{5}\cdot\frac{-\cos(5x)}{5}\right) + c$$

Exercise 2.1.6 *Determine the following indefinite integrals. (partial integration)*

$$\int x \cdot \ln x \,\mathrm{d}x = ? \qquad \int \lg x \,\mathrm{d}x = ?$$

$$\int (x^3 - x^2) \cdot \ln(3x) \,\mathrm{d}x = ?$$

Solution:

$$\int x \cdot \ln x \,\mathrm{d}x = \begin{bmatrix} g(x) = \ln x & f'(x) = x \\ g'(x) = \frac{1}{x} & f(x) = \frac{x^2}{2} \end{bmatrix} = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \,\mathrm{d}x =$$

$$= \frac{x^2}{2} \cdot \ln x - \int \frac{x}{2} \,\mathrm{d}x = \frac{x^2}{2} \cdot \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + c$$

$$\int \lg x \,\mathrm{d}x = \begin{bmatrix} g(x) = \lg x & f'(x) = 1 \\ g'(x) = \frac{1}{x \cdot \ln 10} & f(x) = x \end{bmatrix} = x \cdot \lg x - \int x \cdot \frac{1}{x \cdot \ln 10} \,\mathrm{d}x =$$

$$= x \cdot \lg x - \int \frac{1}{\ln 10} \,\mathrm{d}x = x \cdot \lg x - \frac{1}{\ln 10} \cdot x + c$$

$$\int (x^3 - x^2) \cdot \ln(3x) \,\mathrm{d}x = \begin{bmatrix} g(x) = \ln(3x) & f'(x) = x^3 - x^2 \\ g'(x) = \frac{1}{x} & f(x) = \frac{x^4}{4} - \frac{x^3}{3} \end{bmatrix} =$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3}\right) \cdot \ln(3x) - \int \left(\frac{x^4}{4} - \frac{x^3}{3}\right) \cdot \frac{1}{x} \,\mathrm{d}x =$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3}\right) \cdot \ln(3x) - \int \left(\frac{x^3}{4} - \frac{x^2}{3}\right) \,\mathrm{d}x =$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3}\right) \cdot \ln(3x) - \left(\frac{x^4}{4 \cdot 4} - \frac{x^3}{3 \cdot 3}\right) + c$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3}\right) \cdot \ln(3x) - \left(\frac{x^4}{16} - \frac{x^3}{9}\right) + c$$

Exercise 2.1.7 *Determine the following indefinite integrals. (partial integration)*

$$\int \sin(2x)\cdot\sin(3x)\,\mathrm{d}x = ? \qquad \int \mathrm{e}^{3x}\cdot\cos(2x)\,\mathrm{d}x = ?$$

Solution:

$$\underline{\int \sin(2x)\cdot\underbrace{\sin(3x)}_{g}\,\mathrm{d}x} = \frac{-\cos(2x)}{2}\cdot\sin(3x) - \int\frac{-\cos(2x)}{2}\cdot\underbrace{3\cdot\cos(3x)}_{g'}\,\mathrm{d}x =$$

$$= \frac{-\cos(2x)}{2}\cdot\sin(3x) + \frac{3}{2}\int\cos(2x)\cdot\underbrace{\cos(3x)}_{\text{new } g}\,\mathrm{d}x =$$

$$= \frac{-\cos(2x)}{2}\cdot\sin(3x) + \frac{3}{2}\cdot\left(\frac{\sin(2x)}{2}\cdot\cos(3x) - \int\frac{\sin(2x)}{2}\cdot(-3)\cdot\sin(3x)\,\mathrm{d}x\right) =$$

$$= \frac{-\cos(2x)}{2}\cdot\sin(3x) + \frac{3}{2}\cdot\left(\frac{\sin(2x)}{2}\cdot\cos(3x) + \frac{3}{2}\cdot\int\sin(2x)\cdot\sin(3x)\,\mathrm{d}x\right) =$$

$$= \frac{-\cos(2x)}{2}\cdot\sin(3x) + \frac{3}{4}\cdot\sin(2x)\cdot\cos(3x) + \frac{9}{4}\cdot\underline{\int\sin(2x)\cdot\sin(3x)\,\mathrm{d}x}$$

Thus

$$\int\sin(2x)\cdot\sin(3x)\,\mathrm{d}x =$$

$$= \frac{-\cos(2x)}{2}\cdot\sin(3x) + \frac{3}{4}\cdot\sin(2x)\cdot\cos(3x) + \frac{9}{4}\cdot\int\sin(2x)\cdot\sin(3x)\,\mathrm{d}x$$

which implies that

$$-\frac{5}{4}\cdot\int\sin(2x)\cdot\sin(3x)\,\mathrm{d}x = \frac{-\cos(2x)}{2}\cdot\sin(3x) + \frac{3}{4}\cdot\sin(2x)\cdot\cos(3x) + c.$$

We have that

$$\int\sin(2x)\cdot\sin(3x)\,\mathrm{d}x = -\frac{4}{5}\left(\frac{-\cos(2x)}{2}\cdot\sin(3x) + \frac{3}{4}\cdot\sin(2x)\cdot\cos(3x) + c\right).$$

$$\underline{\int e^{3x} \cdot \underbrace{\cos(2x)}_{g}\, dx} = \frac{e^{3x}}{3} \cdot \cos(2x) - \int \frac{e^{3x}}{3} \cdot \underbrace{(-2)\cdot\sin(2x)}_{g'}\, dx =$$

$$= \frac{e^{3x}}{3} \cdot \cos(2x) + \frac{2}{3}\int e^{3x} \cdot \underbrace{\sin(2x)}_{\text{new } g}\, dx =$$

$$= \frac{e^{3x}}{3} \cdot \cos(2x) + \frac{2}{3} \cdot \left(\frac{e^{3x}}{3} \cdot \sin(2x) - \int \frac{e^{3x}}{3} \cdot 2 \cdot \cos(2x)\, dx\right) =$$

$$= \frac{e^{3x}}{3} \cdot \cos(2x) + \frac{2}{3} \cdot \left(\frac{e^{3x}}{3} \cdot \sin(2x) - \frac{2}{3} \cdot \int e^{3x} \cdot \cos(2x)\, dx\right) =$$

$$= \frac{e^{3x}}{3} \cdot \cos(2x) + \frac{2}{9} \cdot e^{3x} \cdot \sin(2x) - \frac{4}{9} \cdot \underline{\int e^{3x} \cdot \cos(2x)\, dx}$$

Thus

$$\int e^{3x} \cdot \cos(2x)\, dx = \frac{e^{3x}}{3} \cdot \cos(2x) + \frac{2}{9} \cdot e^{3x} \cdot \sin(2x) - \frac{4}{9} \cdot \int e^{3x} \cdot \cos(2x)\, dx$$

$$\frac{13}{9} \cdot \int e^{3x} \cdot \cos(2x)\, dx = \frac{e^{3x}}{3} \cdot \cos(2x) + \frac{2}{9} \cdot e^{3x} \cdot \sin(2x) + c$$

$$\int e^{3x} \cdot \cos(2x)\, dx = \frac{9}{13} \cdot \left(\frac{e^{3x}}{3} \cdot \cos(2x) + \frac{2}{9} \cdot e^{3x} \cdot \sin(2x) + c\right).$$

Exercise 2.1.8 *Determine the following indefinite integrals.*

$$\left(\int g' \cdot f(g)\, dx = F(g) + c, \quad F = \int f\right)$$

Solution: • $\displaystyle\int \frac{81x}{1+81x^2}\, dx = \int 81x \cdot \frac{1}{1+81x^2}\, dx =$

$$= \frac{1}{2} \cdot \int 2 \cdot 81x \cdot \frac{1}{1+81x^2}\, dx = \frac{1}{2} \cdot \ln|1+81x^2| + c$$

$$\left(f(x) = \frac{1}{x}, \quad g(x) = 1 + 81x^2 \right)$$

- $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}\,\mathrm{d}x = \int x^2 \cdot \frac{1}{\sqrt{1-(x^3)^2}}\,\mathrm{d}x = \frac{1}{3} \cdot \int 3x^2 \cdot \frac{1}{\sqrt{1-(x^3)^2}}\,\mathrm{d}x = \frac{1}{3} \cdot \arcsin(x^3) + c$

$$\left(f(x) = \frac{1}{\sqrt{1-x^2}}, \quad g(x) = x^3 \right)$$

- $\displaystyle\int \frac{-x}{\cos^2 x^2}\,\mathrm{d}x = \int -x \cdot \frac{1}{cos^2(x^2)}\,\mathrm{d}x = -\frac{1}{2} \cdot \int 2x \cdot \frac{1}{cos^2(x^2)}\,\mathrm{d}x = -\frac{1}{2} \cdot \mathrm{tg}(x^2) + c$

$$\left(f(x) = \frac{1}{\cos^2 x}, \quad g(x) = x^2 \right)$$

- $\displaystyle\int \frac{1}{x^2 \cdot \sin^2 \frac{1}{x}}\,\mathrm{d}x = -\int -\frac{1}{x^2} \cdot \frac{1}{\sin^2 \frac{1}{x}}\,\mathrm{d}x = -\left(-\mathrm{ctg}\frac{1}{x} \right) + c$

$$\left(f(x) = \frac{1}{\sin^2 x}, \quad g(x) = \frac{1}{x} \right)$$

Exercise 2.1.9 *Determine the following indefinite integral:* $\displaystyle\int \arcsin x\,\mathrm{d}x$.

Solution:

$$\int f' \cdot g = f \cdot g - \int f \cdot g'$$

$$\int \arcsin x\,\mathrm{d}x = \int 1 \cdot \arcsin x\,\mathrm{d}x = \begin{bmatrix} f'(x) = 1 & g(x) = \arcsin x \\ f(x) = x & g'(x) = \frac{1}{\sqrt{1-x^2}} \end{bmatrix} =$$

$$= x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = x \cdot \arcsin x + \frac{1}{2}\int -2x \cdot (1-x^2)^{-\frac{1}{2}}\,\mathrm{d}x =$$

$$= x \cdot \arcsin x + \tfrac{1}{2} \cdot \tfrac{2}{1} \cdot (1-x^2)^{\frac{1}{2}} + c = x \cdot \arcsin x + \sqrt{1-x^2} + c$$

Exercise 2.1.10 *Determine the following indefinite integral:* $\displaystyle\int \frac{2x+3}{x^2+5x+6}\,\mathrm{d}x$.

Solution: The first step is to calculate the roots of the polynomial $P(x) = x^2+5x+6$. Then we have

$$x_{1,2} = \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5 \pm 1}{2},$$

$x_1 = -3$, $x_2 = -2$. P can be written in the form

$$\int \frac{2x+3}{x^2+5x+6}\,\mathrm{d}x = \int \frac{2x+3}{(x+2)(x+3)}\,\mathrm{d}x$$

$$\frac{2x+3}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$2x+3 = A(x+3) + B(x+2),$$

$$2x+3 = x(A+B) + 3A + 2B.$$

Now we have to solve the following system of equations:

$$\begin{aligned} 2 &= A + B \\ 3 &= 3A + 2B \end{aligned}$$

We get that $A = -1$ and $B = 3$. Therefore

$$\int \frac{2x+3}{(x+2)(x+3)}\,\mathrm{d}x = \int \frac{-1}{x+2} + \frac{3}{x+3}\,\mathrm{d}x = -1\ln|x+2| + 3\ln|x+3| + c.$$

Using the identity of the logarithm we have $\int \frac{2x+3}{(x+2)(x+3)}\,\mathrm{d}x = \ln\left|\frac{(x+3)^3}{x+2}\right| + c$.

Exercise 2.1.11 *Determine the following indefinite integral.*

Solution:

- $\displaystyle\int \frac{x+1}{x(x^2+1)}\,\mathrm{d}x$

$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$.

$$x+1 = A(x^2+1) + (Bx+C)x$$
$$x+1 = (A+B)x^2 + Cx + A.$$

So we have that

$$\begin{aligned} A+B &= 0 \\ C &= 1 \\ A &= 1 \end{aligned}$$

The solution of the system is $A=1$, $B=-1$, $C=1$.

$$\int \frac{x+1}{x(x^2+1)}\,\mathrm{d}x = \int \frac{1}{x} + \frac{-x+1}{x^2+1}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x + \int \frac{-x+1}{x^2+1}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x +$$
$$+ \int \frac{-x}{x^2+1}\,\mathrm{d}x + \int \frac{1}{x^2+1}\,\mathrm{d}x = \ln|x| - \frac{1}{2}\ln(x^2+1) + \operatorname{arctg} x + c.$$

- $\displaystyle\int \frac{3}{25+16x^2}\,\mathrm{d}x$

$$\int \frac{3}{25+16x^2}\,\mathrm{d}x = 3\int \frac{1}{25+16x^2}\,\mathrm{d}x = \frac{3}{25}\int \frac{1}{1+\frac{16}{25}x^2}\,\mathrm{d}x = \frac{3}{25}\int \frac{1}{1+\left(\frac{4}{5}x\right)^2}\,\mathrm{d}x =$$
$$= \frac{3}{25}\frac{\operatorname{arctg}\frac{4}{5}x}{\frac{4}{5}} + c = \frac{3}{25}\operatorname{arctg}\left(\frac{4}{5}x\right)\cdot\frac{5}{4} + c = \frac{3}{20}\operatorname{arctg}\left(\frac{4}{5}x\right) + c$$

- $\displaystyle\int \frac{2}{\sqrt{36-16x^2}}\,\mathrm{d}x$

$$\int \frac{2}{\sqrt{36-16x^2}}\,\mathrm{d}x = 2\int \frac{1}{\sqrt{36-16x^2}}\,\mathrm{d}x = \frac{2}{6}\int \frac{1}{\sqrt{1-\frac{16}{36}x^2}}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{4}{6}x\right)^2}}\,\mathrm{d}x =$$
$$= \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{2}{3}x\right)^2}}\,\mathrm{d}x = \frac{1}{3}\frac{\arcsin\left(\frac{2}{3}x\right)}{\frac{2}{3}} + c = \frac{1}{2}\arcsin\left(\frac{2}{3}x\right) + c$$

- $\displaystyle\int sin^2x \cdot cos^2x \,\mathrm{d}x$

$(\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x,\ \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x,\ \sin 2x = 2\sin x \cos x)$

$$\int sin^2x{\cdot}cos^2x \,\mathrm{d}x = \int (sinx{\cdot}cosx)^2 \,\mathrm{d}x = \int (\frac{1}{2}\sin 2x)^2 \,\mathrm{d}x = \int \frac{1}{4}\sin^2 2x \,\mathrm{d}x =$$

$$= \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x \,\mathrm{d}x = \frac{1}{4}\left(\frac{1}{2}x - \frac{1}{2}\frac{\sin 4x}{4}\right) + c$$

- $\displaystyle\int sin^4x \,\mathrm{d}x$

$$\int sin^4x \,\mathrm{d}x = \int (sin^2x)^2 \,\mathrm{d}x = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 \mathrm{d}x = \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x \,\mathrm{d}x =$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x \,\mathrm{d}x =$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x \,\mathrm{d}x = \frac{1}{4}\int -2\cos 2x + \frac{3}{2} + \frac{1}{2}\cos 4x \,\mathrm{d}x =$$

$$= \frac{1}{4}\left(-2\frac{\sin 2x}{2} + \frac{3}{2}x + \frac{1}{2}\frac{\sin 4x}{4}\right) + c$$

Exercise 2.1.12 *Determine the following indefinite integral.*

$$\left(\int f(g) \cdot g'\right) \circ g^{-1} = F, \ \textit{where } F' = f$$

Solution:

- $\displaystyle\int \frac{1}{1+\sqrt{x}} \,\mathrm{d}x$

Because of substitution $\sqrt{x} = t$ we have, that

$$x = t^2, \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = (t^2)' = 2t,$$

and the integral can be written in the form

$$\int \frac{1}{1+\sqrt{x}}\,\mathrm{d}x = \int \frac{1}{1+t}\,2t\,\mathrm{d}t = 2\int \frac{t}{1+t}\,\mathrm{d}t = 2\int \frac{t+1-1}{1+t}\,\mathrm{d}t =$$
$$= 2\int 1 - \frac{1}{1+t}\,\mathrm{d}t = 2(t - \ln|t+1|) + c = 2\left(\sqrt{x} - \ln(\sqrt{x}+1)\right) + c.$$

- $\displaystyle\int \frac{e^x}{1+e^{2x}}\,\mathrm{d}x$

Because of substitution $\mathrm{e}^x = t$ we have, that

$$x = \ln t, \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = (\ln t)' = \frac{1}{t},$$

and the integral can be written in the form

$$\int \frac{\mathrm{e}^x}{1+\mathrm{e}^{2x}}\,\mathrm{d}x = \int \frac{t}{1+t^2}\cdot\frac{1}{t}\,\mathrm{d}t = \int \frac{1}{1+t^2}\,\mathrm{d}t = \operatorname{arctg} t + c = \operatorname{arctg}\mathrm{e}^x + c.$$

- $\displaystyle\int \frac{\sqrt{x+1}}{x+2}\,\mathrm{d}x$

Because of substitution $\sqrt{x+1} = t$ we have, that

$$x = t^2 - 1, \quad \text{and} \quad \frac{dx}{dt} = 2t.$$

and the integral can be written in the form

$$\int \frac{\sqrt{x+1}}{x+2}\,\mathrm{d}x = \int \frac{t}{t^2+1}\cdot 2t\,\mathrm{d}t = \int \frac{2t^2}{t^2+1}\,\mathrm{d}t = 2\int \frac{t^2}{t^2+1}\,\mathrm{d}t =$$
$$= 2\int \frac{t^2+1-1}{t^2+1}\,\mathrm{d}t = 2\int 1 - \frac{1}{t^2+1}\,\mathrm{d}t = 2\left(t - \operatorname{arctg} t\right) + c =$$
$$= 2\left(\sqrt{x+1} - \operatorname{arctg}\sqrt{x+1}\right) + c.$$

2.2 Riemann integral (Definit integral), improper integral

Exercise 2.2.1 *Calculate the following integrals:*

$$\text{(I.)} \int_0^{\frac{\pi}{4}} \mathrm{tg} x \, \mathrm{d}x \qquad \text{(II.)} \int_0^1 x \cdot e^{2x} \, \mathrm{d}x$$

Solution: (I.) We use the following rule: $\int \frac{g'}{g} \, \mathrm{d}x = \ln|g| + c$; after that we apply the Newton-Leibniz formula:

$$\int_0^{\frac{\pi}{4}} \mathrm{tg} x \, \mathrm{d}x = -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} \, \mathrm{d}x = -[\ln|\cos x|]_0^{\frac{\pi}{4}} = -(\ln|\cos \frac{\pi}{4}| - \ln|\cos 0|) =$$

$$= -\left(\ln \frac{\sqrt{2}}{2} - \ln 1\right) = -\ln \frac{\sqrt{2}}{2} \approx 0.347$$

(II.) By the help of the partial integration:

$$\int_a^b f' \cdot g = [f \cdot g]_a^b - \int_a^b f \cdot g'$$

$$\int_0^1 x \cdot \mathrm{e}^{2x} \, \mathrm{d}x = \left[x \cdot \frac{\mathrm{e}^{2x}}{2}\right]_0^1 - \int_0^1 \frac{\mathrm{e}^{2x}}{2} \, \mathrm{d}x = \left[x \cdot \frac{\mathrm{e}^{2x}}{2}\right]_0^1 - \left[\frac{\mathrm{e}^{2x}}{4}\right]_0^1 =$$

$$= \frac{\mathrm{e}^2}{2} - \frac{\mathrm{e}^2}{4} + \frac{1}{4} = \frac{\mathrm{e}^2 + 1}{4}.$$

Exercise 2.2.2 *Calculate the following improper integrals:*

$$\text{(I.)} \int_2^{\infty} \frac{1}{x^3} \, \mathrm{d}x \quad \text{(II.)} \int_{-\infty}^{-2} \frac{1}{x^3} \, \mathrm{d}x \quad \text{(III.)} \int_0^1 \ln x \, \mathrm{d}x$$

Solution: (I.) Using the definition:

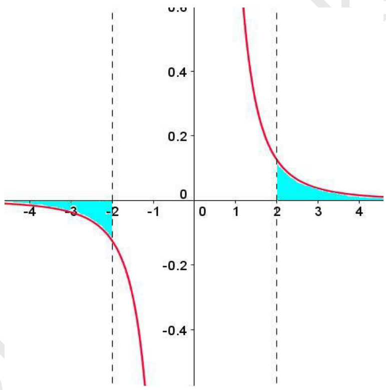
$$\int_2^{\infty} \frac{1}{x^3}\,\mathrm{d}x = \lim_{c\to\infty} \int_2^{c} \frac{1}{x^3}\,\mathrm{d}x = \lim_{c\to\infty} \int_2^{c} x^{-3}\,\mathrm{d}x =$$

$$= \lim_{c\to\infty} \left[-\frac{1}{2x^2}\right]_2^c = \lim_{c\to\infty} \left(-\frac{1}{2c^2} + \frac{1}{8}\right) = \frac{1}{8}.$$

(II.) Using the definition:

$$\int_{-\infty}^{-2} \frac{1}{x^3}\,\mathrm{d}x = \lim_{c\to-\infty} \int_c^{-2} \frac{1}{x^3}\,\mathrm{d}x = \lim_{c\to-\infty} \int_c^{-2} x^{-3}\,\mathrm{d}x =$$

$$= \lim_{c\to-\infty} \left[-\frac{1}{2x^2}\right]_c^{-2} = \lim_{c\to-\infty} \left(-\frac{1}{2\cdot(-2)^2} + \frac{1}{2c^2}\right) = -\frac{1}{8}.$$



(III.)

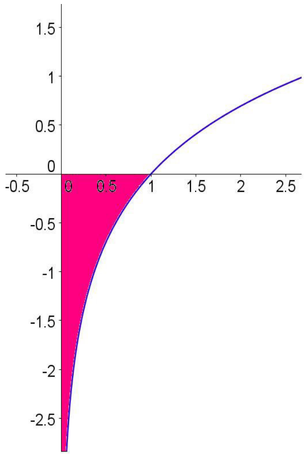
$$\int_0^1 \ln x\,\mathrm{d}x = \lim_{c\to0+0} \int_c^1 \ln x\,\mathrm{d}x \overbrace{=}^{\star} \lim_{c\to0+0} \left([x\cdot \ln x]_c^1 - \int_c^1 1\,\mathrm{d}x\right) =$$

$$= \lim_{c\to0+0} \left([x\cdot \ln x]_c^1 - [x]_c^1\right) = \lim_{c\to0+0} (1\cdot \ln 1 - c\cdot \ln c - (1-c)) =$$

$$= \lim_{c\to0+0} (-c\cdot \ln c - 1 + c) \overbrace{=}^{\star\star} 0 - 1 + 0 = -1$$

$\star$ We use the partial integration rule.

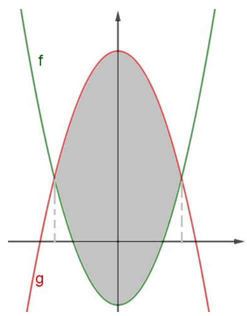
$\star\star$ $\lim\limits_{c\to 0+0}(-c\cdot \ln c) = \lim\limits_{c\to 0+0}\frac{-\ln c}{\frac{1}{c}} \overbrace{=}^{L'H} \lim\limits_{c\to 0+0}\frac{-\frac{1}{c}}{-\frac{1}{c^2}} = \lim\limits_{c\to 0+0} c = 0$



2.3 Applications

Exercise 2.3.1 *Calculate the area of the domain enclosed by* $f(x) = x^2 - 2$ *and* $g(x) = 6 - x^2$.

Solution: At first we give the schematic graph of the functions. The second step is to solve the equation $f(x) = g(x)$.



$$f(x) = g(x) \Leftrightarrow \; x^2 - 2 = 6 - x^2 \; \Leftrightarrow \; 2x^2 = 8 \; \Leftrightarrow \; x^2 = 4 \; \Leftrightarrow \; x = \pm 2$$

Thus the area:

$$T = \int_{-2}^{2} 6 - x^2 \, \mathrm{d}x - \int_{-2}^{2} x^2 - 2 \, \mathrm{d}x = \int_{-2}^{2} 6 - x^2 - (x^2 - 2) \, \mathrm{d}x = \int_{-2}^{2} 8 - 2x^2 \, \mathrm{d}x =$$

$$= \left[8x - \frac{2x^3}{3} \right]_{-2}^{2} = \left(8 \cdot 2 - \frac{2 \cdot 2^3}{3} \right) - \left(8 \cdot (-2) - \frac{2 \cdot (-2)^3}{3} \right) =$$

$$= 16 - \frac{16}{3} + 16 - \frac{16}{3} = 32 - \frac{32}{3} = \frac{96 - 32}{3} = \frac{64}{3}.$$

Exercise 2.3.2 *Let T be the domain (solid) obtained by rotating the curve of $f(x) = \frac{1}{\sqrt{1+x^2}}$, $\;x \in [-1, 1]$. Calculate the volume of T.*

Solution:

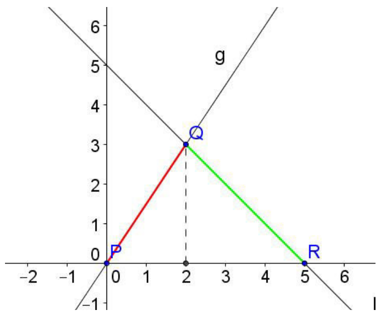
$$V = \pi \cdot \int_a^b f^2(x) \, \mathrm{d}x$$

$$V = \pi \cdot \int_{-1}^{1} \left(\frac{1}{\sqrt{1 + x^2}} \right)^2 \mathrm{d}x = \pi \cdot \int_{-1}^{1} \frac{1}{1 + x^2} \, \mathrm{d}x = \pi \cdot [\mathrm{arctg}(x)]_{-1}^{1} =$$

$$= \pi \cdot (\mathrm{arctg}(1) - \mathrm{arctg}(-1)) = \pi \cdot \left(\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right) = \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{2} \approx 4.93$$

Exercise 2.3.3 *Let T be the domain (solid) obtained by rotating the rectangle wich determined by its vertexes: (0,0), (2,3), (5,0) Calculate the volume of T.*

Solution:



The line which passes through on the points (0,0) and (2,3): $g(x) = \frac{3}{2} \cdot x$,
The line which passes through on the points (2,3) and (5,0): $l(x) = -x + 5$.

$$V = V_1 + V_2 = \pi \cdot \int_0^2 g^2(x)\,\mathrm{d}x + \pi \cdot \int_2^5 l^2(x)\,\mathrm{d}x, \text{ így}$$

$$V = \pi \left(\int_0^2 \frac{9}{4} \cdot x^2 \,\mathrm{d}x + \int_2^5 (5-x)^2 \,\mathrm{d}x \right) =$$

$$= \pi \cdot \left(\left[\frac{9}{4} \cdot \frac{x^3}{3} \right]_0^2 + \left[\frac{(5-x)^3}{3 \cdot (-1)} \right]_2^5 \right) = \pi \cdot \left[\left(\frac{9}{4} \cdot \frac{8}{3} - 0 \right) + \left(0 - \frac{27}{-3} \right) \right] =$$

$= \pi \cdot (6 + 9) = 15\pi$.

Remark: By the volume of the cones:

the radius of the circle: r, the height of the cone: m, then $V = \frac{r^2 \pi m}{3}$.
$V = \frac{3^2\pi \cdot 2}{3} + \frac{3^2 \pi \cdot 3}{3} = 15\pi$.

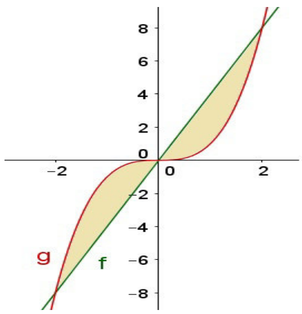
Exercise 2.3.4 *Calculate the area enclosed by* $f(x) = 4x$ $(x \in \mathbb{R})$ *and* $g(x) = x^3$ $(x \in \mathbb{R})$.

Solution: Calculation of the borders of the integration:

$4x = x^3 \;\Leftrightarrow\; 0 = x^3 - 4x \;\Leftrightarrow\; 0 = x \cdot (x^2 - 4) \;\Leftrightarrow\; x_1 = -2,\; x_2 = 0,\; x_3 = 2$

Because $g(x) \geq f(x)$ if $x \in [-2, 0]$ and $f(x) \geq g(x)$ if $[0, 2]$, thus the investigated area:

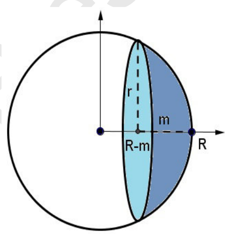
$$A = \int_{-2}^{0} g(x) - f(x)\,dx + \int_0^2 f(x) - g(x)\,dx \qquad \underbrace{=}_{\text{f and g is also odd function}}$$



$$= 2 \cdot \int_0^2 f(x) - g(x)\, dx = \int_0^2 4x - x^3\, dx = 2 \cdot \left[2x^2 - \frac{x^4}{4} \right]_0^2 =$$

$$= 2 \cdot \left(2 \cdot 2^2 - \frac{2^4}{4} - \left(\underbrace{2 \cdot 0^2 - \frac{0^4}{4}}_{0} \right) \right) = 2 \cdot (8 - 4) = 8$$

Exercise 2.3.5 *Give the formula for the volume of spherical segments.*



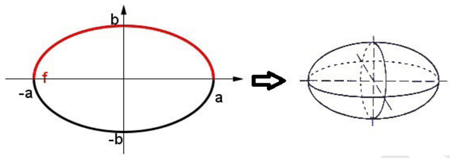
Solution. $f(x) = \sqrt{R^2 - x^2}, \quad x \in [R - m, R]$
$f'(x) = \frac{1}{2\sqrt{R^2-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{R^2-x^2}}$

$$A = r^2\pi + 2\pi \int_{R-m}^{R} \sqrt{R^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, \mathrm{d}x =$$

$$= r^2\pi + 2\pi \int_{R-m}^{R} \sqrt{R^2 - x^2} \cdot \sqrt{\frac{R^2}{R^2 - x^2}}\, \mathrm{d}x = r^2\pi + 2\pi \cdot \int_{R-m}^{R} R\, \mathrm{d}x =$$

$$= r^2\pi + 2\pi \cdot [R \cdot x]_{R-m}^{R} = r^2\pi + 2\pi \cdot \Big(R^2 - R(R-m)\Big) = r^2\pi + 2\pi Rm$$

Exercise 2.3.6 *By rotating the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *around the x-axis we have a spheroid. Determine the volume of the spheroid.*



$$y^2 = b^2 \cdot \left(1 - \tfrac{x^2}{a^2}\right) \;\Rightarrow\; y = b \cdot \sqrt{1 - \tfrac{x^2}{a^2}} \;\Rightarrow\; f(x) = b \cdot \sqrt{1 - \tfrac{x^2}{a^2}}$$

$$V = \pi \cdot \int_{-a}^{a} b^2 \cdot \left(1 - \frac{x^2}{a^2}\right) \mathrm{d}x = b^2 \cdot \pi \cdot \left[x - \frac{x^3}{3a^2}\right]_{-a}^{a} =$$

$$= b^2 \cdot \pi \cdot \left(\underbrace{a - \frac{a^3}{3a^2}}_{a-\frac{a}{3}} - \left(\underbrace{-a - \frac{(-a)^3}{3a^2}}_{-a+\frac{a}{3}}\right)\right) = \frac{4}{3} a \cdot b^2 \cdot \pi$$

3

Differential equations

3.1 Directly integrable differential equations

Exercise 3.1.1 *Show that $y(x) = cx^2$ are solutions of the equation*

$$xy'(x) = 2y(x)$$

for all $c \in \mathbb{R}$.

Solution. Left-hand side:

$$xy'(x) = x(cx^2)' = x \cdot 2cx = 2cx^2,$$

Right-hand side:

$$2y(x) = 2cx^2,$$

thus the left-hand side and the right-hand side of the equation is the same for all$c \in \mathbb{R}$, which means that $y(x) = cx^2$ are soltutions of the differential equation.

Exercise 3.1.2 *Solve the following initial value problem: $y'(x) = \cos 2x$, $y(0) = 1$.*

Solution. By integration we get the solutions of the given DE:

$$y(x) = \frac{\sin 2x}{2} + c \quad (c \in \mathbb{R})$$

$y(0) = 1$, thus

$$1 = y(0) = \frac{\sin 0}{2} + c = c,$$

i.e. $c = 1$. The solution of the initial value problem:

$$y(x) = \frac{\sin 2x}{2} + 1.$$

Exercise 3.1.3 *Solve the following initial value problem:*

$$y'(x) = \frac{x}{x^2+1}, \quad y(0) = 2$$

Solution. By integration we get the solutions of the given DE:

$$y(x) = \int \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\ln(x^2+1) + c$$

$(c \in \mathbb{R})$.

$y(0) = 2$, more precisely $2 = y(0) = \frac{1}{2}\ln 1 + c = c$, thus$c = 2$. The solution of the initial value problem:

$$y(x) = \frac{1}{2}\ln(x^2+1) + 2.$$

Exercise 3.1.4 *Solve the following initial value problem:*

$$y'(x) = x \cdot \mathrm{e}^x, \qquad y(0) = 3$$

Check your result.

Solution. By integration we get the solutions of the given DE:

$$y(x) = \int x \cdot \mathrm{e}^x \,\mathrm{d}x = x \cdot \mathrm{e}^x - \int \mathrm{e}^x \,\mathrm{d}x = x \cdot \mathrm{e}^x - \mathrm{e}^x + c.$$

On the other hand $3 = y(0) = -\mathrm{e}^0 + c = -1 + c$, thus $c = 4$. The solution of the initial value problem:

$$y(x) = (x-1)\mathrm{e}^x + 4.$$

Check:

$$\left((x-1)\mathrm{e}^x + 4\right)' = \mathrm{e}^x + (x-1)\mathrm{e}^x = x\mathrm{e}^x, \quad y(0) = -1 \cdot \mathrm{e}^0 + 4 = 3.$$

3.2 Separable differential equations

Let I and J be open intervals, $g\colon I \to \mathbb{R}$ and $h\colon J \to \mathbb{R}$ be continuous functions. Equation

$$y'(x) = g(x) \cdot h(y), \qquad h \neq 0$$

is called a separable differential equation. There exists a solution $y\colon I \to \mathbb{R}$ of equation

$$y'(x) = g(x) \cdot h(y)$$

if and only if equality

$$\int \frac{1}{h(y)} \,\mathrm{d}y = \int g(x) \,\mathrm{d}x$$

holds.

Exercise 3.2.1 *The change in a bacteria population is proportional to the population. How many bacterias were at the end of the 24-th hours if during 48 hours the numbers of the bacterias have increased from 100 to 1000?*

Solution. If $N(t)$ notes the numbers of the a bacterias at the moment t, then

$$\dot{N}(t) = \lambda \cdot N(t), \quad N(0) = 100.$$

The solution of this initial value problem: $N(t) = 100 \cdot e^{\lambda \cdot t}$.

On the other hand $N(48) = 1000$, thus $1000 = N(48) = 100 \cdot \mathrm{e}^{48\lambda}$. Therefore

$$10 = \mathrm{e}^{48\lambda} \Rightarrow \lambda = \frac{\ln 10}{48}.$$

Thus

$$N(24) = 100 \cdot \mathrm{e}^{24 \cdot \frac{\ln 10}{48}} = 100 \cdot \mathrm{e}^{\frac{1}{2}\ln 10} = 100 \cdot \mathrm{e}^{\ln \sqrt{10}} = 100 \cdot \sqrt{10} \approx 316,$$

thus approximately there were 316 pieces bacteria at the end of the 24-th hours.

Exercise 3.2.2 *Solve the following initial value problem:*

$$y'(x) = xy(x), \quad y(0) = 1$$

Solution. The investigated equation is a separable DE, $g(x) = x$, $h(y) = y$, thus equality

$$\int \frac{1}{y} \,\mathrm{d}y = \int x \,\mathrm{d}x$$

holds. By integration we get that

$$\ln |y| = \frac{x^2}{2} + c$$

which from we have

$$y = \pm \mathrm{e}^{\frac{x^2}{2} + c} = \pm \mathrm{e}^{\frac{x^2}{2}} \cdot \mathrm{e}^{c} = C \cdot \mathrm{e}^{\frac{x^2}{2}}. \ \ (C \in \mathbb{R})$$

On the other hand $y(0) = 1$, thus $1 = y(0) = C \cdot \mathrm{e}^0 = C$, i.e. the solution of the initial value problem is $y(x) = \mathrm{e}^{\frac{x^2}{2}}$.

Exercise 3.2.3 *Solve the following DE:*

$$y' = \frac{\sin x}{y^2}$$

Solution. As

$$y' = \frac{\sin x}{y^2} = \sin x \cdot \frac{1}{y^2},$$

thus $g(x) = \sin x$, $h(y) = 1/y^2$. Using equation $\int \frac{1}{h(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$ we get that

$$\int y^2\,\mathrm{d}y = \int \sin x\,\mathrm{d}x.$$

which from it follows that

$$\frac{y^3}{3} = -\cos x + c, \quad \text{i.e.} \quad y(x) = \sqrt[3]{3(c - \cos x)}.$$

Exercise 3.2.4 *Solve the following DE:* $y' = -x \cdot \mathrm{e}^y$

Solution. Similarly as above $g(x) = -x$, $h(y) = \mathrm{e}^y$ ($\int \frac{1}{h(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$), i.e.

$$\int \frac{1}{\mathrm{e}^y}\,\mathrm{d}y = \int -x\,\mathrm{d}x.$$

On the left hand side we have e^{-y}, thus we can use that $\int f(ay+b)\mathrm{d}y = \frac{F(ax+b)}{a} + c$ as follows:

$$-\mathrm{e}^{-y} = -\frac{x^2}{2} + c$$

Multiplied by -1 we can write that $\mathrm{e}^{-y} = \frac{x^2}{2} + C$. Using the inverse of the exponential function in both side and multiplied by -1 we get that

$$y(x) = -\ln\left(\frac{x^2}{2} + C\right).$$

Exercise 3.2.5 *Solve the following DE:*

$$y' = \operatorname{tg} x \cdot \operatorname{tg} y$$

Solution. The investigated equation is a separable DE, $g(x) = \operatorname{tg} x$, $h(y) = \operatorname{tg} y$, thus

$$\int \frac{1}{\operatorname{tg} y}\,\mathrm{d}y = \int \operatorname{tg} x\,\mathrm{d}x$$

On the left hand side we can use that $\int f'/f = \ln|f| + c$:

$$\int \frac{1}{\operatorname{tg} y}\,\mathrm{d}y = \int \frac{\cos y}{\sin y}\,\mathrm{d}y = \ln|\sin y| + c_1.$$

We can integrate on the right hand side similarly and we get that:

$$\int \operatorname{tg} x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x = -\ln|\cos x| + c_2.$$

Deriving notation $c = c_2 - c_1$ we have that

$$\ln|\sin y| = -\ln|\cos x| + c.$$

From this equality we have to express y:

$$\begin{aligned}
\ln|\sin y| &= -\ln|\cos x| + c \\
\sin y &= \pm \mathrm{e}^{-\ln|\cos x| + c} \\
\sin y &= \pm \mathrm{e}^{-\ln|\cos x|} \cdot \mathrm{e}^{c} \\
\sin y &= \mathrm{e}^{\ln|\cos x|^{-1}} \cdot C \\
\sin y &= \frac{1}{\cos x} \cdot C \\
y(x) &= \arcsin\left(\frac{C}{\cos x}\right).
\end{aligned}$$

3.3 First order linear inhomogeneous DE

The complementary solution of equation $y'(x) + a(x)y(x) = r(x)$:

1. the complementary solution y_c of the homogeneous part $(y'(x) + a(x)y(x) = 0)$:
$$\left(y_c(x) = C \cdot e^{-\int a(x)\mathrm{d}x}\right)$$

2. a particular solution y_p of the inhomogeneous part $(y'(x)+a(x)y(x) = r(x))$:
$$\left(\text{with the method of the variation: } y_p(x) = C(x) \cdot e^{-\int a(x)\mathrm{d}x}\right)$$

3. the general solution of the inhomogeneous DE: $y(x) = y_c(x) + y_p(x)$

Exercise 3.3.1 *Solve the following DE:*
$$y'(x) = 2x \cdot y(x) + \mathrm{e}^{x^2}$$

Solution This is a first order linear inhomogeneous DE.

1. the complementary solution y_c of the homogeneous part $y'(x)-2xy(x) = 0$:
$$y_c(x) = C \cdot e^{-\int -2x\mathrm{d}x} = C \cdot e^{x^2}$$

2. a particular solution y_p of the inhomogeneous part: $y_p(x) = C(x) \cdot \mathrm{e}^{x^2}$,thus equation
$$(C(x) \cdot \mathrm{e}^{x^2})' = 2x \cdot C(x) \cdot \mathrm{e}^{x^2} + \mathrm{e}^{x^2}$$
holds.

By derivation on the left-hand side we have that
$$C'(x) \cdot \mathrm{e}^{x^2} + C(x) \cdot \mathrm{e}^{x^2} \cdot 2x = 2x \cdot C(x) \cdot \mathrm{e}^{x^2} + \mathrm{e}^{x^2},$$

(the term contained $C(x)$ always vanishes) i.e.

$$C'(x) \cdot \mathrm{e}^{x^2} = \mathrm{e}^{x^2},$$

which from it follows that $C'(x) = 1$. So $C(x) = x$. (It is enough to give one primitive function, because we want to give one solution of the inhomogeneous DE.) Thus $y_p(x) = x \cdot \mathrm{e}^{x^2}$.

3. the general solution of the inhomogeneous DE: $\underline{\underline{y(x)}} = y_c(x) + y_p(x) = C \cdot \mathrm{e}^{x^2} + x \cdot \mathrm{e}^{x^2} = \underline{\underline{(C+x) \cdot \mathrm{e}^{x^2}}}$.

Remark 3.3.1 $y' - 2xy = 0 \;\Rightarrow\; y' = 2xy$ *as a separable DE:* $g(x) = 2x$, $h(y) = y$.

Exercise 3.3.2 *Solve the following initial value problem:*

$$y'(x) = y(x) + x, \quad y(0) = 1$$

Solution. This is a first order linear inhomogeneous DE.

1. the complementary solution y_c of the homogeneous part $y'(x) - y(x) = 0$:

$$y_c(x) = C \cdot e^{-\int -1 \mathrm{d}x} = C \cdot e^x$$

2. a particular solution y_p of the inhomogeneous part: $y_p(x) = C(x) \cdot \mathrm{e}^x$, thus equality

$$(C(x) \cdot \mathrm{e}^x)' = C(x) \cdot \mathrm{e}^x + x$$

holds. By derivation on the left-hand side we have that

$$C'(x) \cdot \mathrm{e}^x + C(x) \cdot \mathrm{e}^x = C(x) \cdot \mathrm{e}^x + x,$$

i.e.

$$C'(x) \cdot \mathrm{e}^x = x \Rightarrow C'(x) = \frac{x}{\mathrm{e}^x} = x \cdot \mathrm{e}^{-x}.$$

Using the rule $\int f' \cdot g = f \cdot g - \int f \cdot g'$ we get that

$$C(x) = \int x \cdot \mathrm{e}^{-x} \,\mathrm{d}x = -x \cdot \mathrm{e}^{-x} + \int \mathrm{e}^{-x} \,\mathrm{d}x = -x \cdot \mathrm{e}^{-x} - \mathrm{e}^{-x} = (-x-1) \cdot \mathrm{e}^{-x}$$

So $y_p(x) = (-x-1)\mathrm{e}^{-x} \cdot \mathrm{e}^x = -x - 1$.

3. the general solution of the inhomogeneous DE:

$$y(x) = y_c(x) + y_p(x) = C \cdot \mathrm{e}^x - x - 1$$

4. the solution of the initial value problem: $1 = y(0) = C - 1 \;\Rightarrow\; C = 2$, thus $\underline{\underline{y(x) = 2\mathrm{e}^x - x - 1}}$.

Remark 3.3.2 $y' - y = 0 \quad \Rightarrow \quad y' = y$ *as a separable DE:* $g(x) = 1$, $h(y) = y$.

Exercise 3.3.3 *Solve the following DE:*

$$u'(t) = \mathrm{e}^t - u(t)$$

Solution.

1. the complementary solution u_c of the homogeneous part $u'(t) + u(t) = 0$:
$$u_c(t) = C \cdot e^{-\int 1 \mathrm{d}t} = C \cdot e^{-t}$$

2. a particular solution u_p of the inhomogeneous part: $u_p(t) = C(t) \cdot \mathrm{e}^{-t}$, thus
$$(C(t) \cdot \mathrm{e}^{-t})' = \mathrm{e}^t - C(t) \cdot \mathrm{e}^{-t}.$$

   By derivation on the left-hand side we have that
$$C'(t) \cdot \mathrm{e}^{-t} - C(t) \cdot \mathrm{e}^{-t} = \mathrm{e}^t - C(t) \cdot \mathrm{e}^{-t}$$

i.e.
$$C'(t) \cdot \mathrm{e}^{-t} = \mathrm{e}^t \Rightarrow C'(t) = \mathrm{e}^{2t} \Rightarrow C(t) = \frac{\mathrm{e}^{2t}}{2}.$$

Thus
$$u_p(t) = \frac{\mathrm{e}^{2t}}{2} \cdot \mathrm{e}^{-t} = \frac{\mathrm{e}^t}{2}.$$

3. the general solution of the inhomogeneous DE:

$$\underline{\underline{u(t)}} = u_c(t) + u_p(t) = \underline{\underline{C \cdot e^{-t} + \frac{\mathrm{e}^t}{2}}}.$$

Remark 3.3.3 $u' + u = 0 \;\Rightarrow\; u' = -u$ *as a separable DE:* $g(t) = -1$, $h(u) = u$.

Exercise 3.3.4 *Solve the following initial value problem*

$$(t^2 - 1) \cdot x'(t) + 2t \cdot x(t) = 1, \quad x(2) = 4$$

1. the complementary solution x_c of the homogeneous part $(t^2 - 1) \cdot x'(t) + 2t \cdot x(t) = 0$:

$$x_c(t) = C \cdot e^{-\int \frac{2t}{t^2-1}\mathrm{d}t} = C \cdot e^{-ln|t^2-1|} = C \cdot \frac{1}{t^2 - 1} = \frac{C}{t^2 - 1}$$

2. a particular solution x_p of the inhomogeneous part: $x_p(t) = \frac{C(t)}{t^2-1}$, thus

$$(t^2 - 1) \cdot \left(\frac{C(t)}{t^2 - 1}\right)' + 2t \cdot \frac{C(t)}{t^2 - 1} = 1$$

$$(t^2 - 1) \cdot \left(\frac{C'(t) \cdot (t^2 - 1) - C(t) \cdot 2t}{(t^2 - 1)^2}\right) + 2t \cdot \frac{C(t)}{t^2 - 1} = 1$$

i.e.

$$C'(t) = 1 \Rightarrow C(t) = t.$$

So

$$x_p(t) = \frac{t}{t^2 - 1}.$$

3. the general solution of the inhomogeneous DE:

$$x(t) = x_c(t) + x_p(t) = \frac{C + t}{t^2 - 1}.$$

4. the solution of the initial value problem: $4 = x(2) = \frac{C+2}{2^2-1}$, $\Rightarrow$ $C = 10$, thus $\underline{\underline{y(x) = \frac{10+t}{t^2-1}}}$.

3.4 Second order linear DE

Let $p, q \in \mathbb{R}$ and consider equation

$$(\circ) \qquad y''(x) + p \cdot y'(x) + q \cdot y(x) = 0.$$

It can be shown that $y(x) = e^{\lambda \cdot x}$ is a solution of $(\circ)$ if and only if

$$\lambda^2 + p \cdot \lambda + q = 0.$$

This equation is the so-called auxiliary equation of equation $(\circ)$. As a quadratic equation always has solutions on the set of the complex numbers, therefore $(\circ)$ always has solutions. We have three possible cases:

I. the auxiliary equation has two different real roots: λ_1 and λ_2. Then the complementary solution of $(\circ)$:

$$y(x) = c_1 \cdot e^{\lambda_1 \cdot x} + c_2 \cdot e^{\lambda_2 \cdot x}.$$

II. the auxiliary equation has one real root: λ. Then the complementary solution of $(\circ)$:

$$y(x) = c_1 \cdot e^{\lambda \cdot x} + c_2 \cdot x \cdot e^{\lambda \cdot x}.$$

(It is easy to check that $W(x) \neq 0$.)

III. the auxiliary equation has two complex roots: $\lambda_1 = \alpha + \beta \cdot i$ and $\lambda_2 = \alpha - \beta \cdot i$. Then the complementary solution of $(\circ)$:

$$y(x) = c_1 \cdot e^{\alpha \cdot x} \cdot \cos(\beta \cdot x) + c_2 \cdot e^{\alpha \cdot x} \cdot \sin(\beta \cdot x)$$

Exercise 3.4.1 *Solve the following initial value problem:*

$$y''(x) + 7y'(x) + 10y(x) = 0, \quad y(0) = 2, \; y'(0) = 5$$

Solution:
the auxiliary equation of our differential equation: $\lambda^2 + 7\lambda + 10 = 0$

$(a\lambda^2 + b\lambda + c = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-b\pm\sqrt{D}}{2a}$, where $D = b^2 - 4ac)$

$$D = 7^2 - 4 \cdot 10 = 49 - 40 = 9 > 0,$$

so we have two real roots: $\lambda_{1,2} = \frac{-7\pm\sqrt{D}}{2} = \frac{-7\pm 3}{2}$, that is $\lambda_1 = -5$ and $\lambda_2 = -2$.
The general solution of the differential equation:

$$y(x) = c_1\mathrm{e}^{-5x} + c_2\mathrm{e}^{-2x} \qquad (c_1, c_2 \in \mathbb{R}).$$

Now we use the initial assumptions to solve the initial value problem (these are $y(0) = 2,\ y'(0) = 5$).

Because of the first assumption

$$2 = y(0) = c_1\mathrm{e}^{-5\cdot 0} + c_2\mathrm{e}^{-2\cdot 0} = c_1 + c_2.$$

On the other hand, the form of a solution of the differential equation is $y(x) = c_1\mathrm{e}^{-5x} + c_2\mathrm{e}^{-2x}$, thus we can write

$$y'(x) = -5c_1\mathrm{e}^{-5x} - 2c_2\mathrm{e}^{-2x}.$$

Because of the second assumption

$$5 = y'(0) = -5c_1\mathrm{e}^{-5\cdot 0} - 2c_2\mathrm{e}^{-2\cdot 0} = -5c_1 - 2c_2.$$

So we get that the constants c_1 and c_2 satisfy the following system of equations:

$$\begin{aligned} c_1 + \ \ c_2 &= 2 \\ -5c_1 - 2c_2 &= 5 \end{aligned}$$

The solution of the system: $c_1 = -3$, $c_2 = 5$. Thus the solution of the initial value problem:

$$y(x) = -3\mathrm{e}^{-5x} + 5\mathrm{e}^{-2x}.$$

Exercise 3.4.2 *Solve the following initial value problem:*

$$y''(x) + 2y'(x) + y(x) = 0, \quad y(0) = 2,\ y'(0) = -1$$

Solution:

the auxiliary equation of our differential equation: $\lambda^2 + 2\lambda + 1 = 0$
$D = 2^2 - 4 = 4 - 4 = 0$, thus we have one real solution. Because of

$$0 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$$

we get that $\lambda = -1$.
The general solution of the differential equation:

$$y(x) = c_1\mathrm{e}^{-x} + c_2 x\mathrm{e}^{-x} \qquad (c_1, c_2 \in \mathbb{R}).$$

$$2 = y(0) = c_1\mathrm{e}^{-0} + c_2 \cdot 0 \cdot \mathrm{e}^{-0} = c_1,$$

$$y(x) = c_1\mathrm{e}^{-x} + c_2 x\mathrm{e}^{-x} \quad \Rightarrow \quad y'(x) = -c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-x} - c_2 x\mathrm{e}^{-x}$$

$$-1 = y'(0) = -c_1\mathrm{e}^{-0} + c_2\mathrm{e}^{-0} - c_2 \cdot 0 \cdot \mathrm{e}^{-0} = -c_1 + c_2.$$

So we get that the constants c_1 and c_2 satisfy the following system of equations:

$$\begin{aligned} c_1 \quad\;\; &= 2 \\ -c_1 + c_2 &= -1 \end{aligned}$$

The solution of the system: $c_1 = 2$, $c_2 = 1$. Thus the solution of the initial value problem:

$$y(x) = 2\mathrm{e}^{-x} + x\mathrm{e}^{-x}.$$

Exercise 3.4.3 *Solve the following initial value problem:*

$$y''(x) + 4y'(x) + 5y(x) = 0, \quad y(0) = 1,\ y'(0) = 2$$

Solution: the auxiliary equation of our differential equation: $\lambda^2+4\lambda+5=0$ $D=4^2-5\cdot 4=16-20=-4$, so we have complex roots: $\lambda_1=\alpha+\beta\cdot i$ and $\lambda_2=\alpha-\beta\cdot i$.
$\lambda_{1,2}=\frac{-4\pm\sqrt{D}}{2}=\frac{-4\pm\sqrt{-4}}{2}=\frac{-4\pm 2i}{2}=-2\pm i$, so $\alpha=-2$, $\beta=1$.
The general solution of the differential equation:

$$y(x)=c_1\mathrm{e}^{-2x}\cos x+c_2\mathrm{e}^{-2x}\sin x \qquad (c_1,c_2\in\mathbb{R}).$$

$$1=y(0)=c_1\mathrm{e}^0\cos 0+c_2\mathrm{e}^0\sin 0=c_1,\ \text{thus } c_1=1$$

because of $y(x)=c_1\mathrm{e}^{-2x}\cos x+c_2\mathrm{e}^{-2x}\sin x$ we get that

$$y'(x)=-2c_1\mathrm{e}^{-2x}\cos x-c_1\mathrm{e}^{-2x}\sin x-2c_2\mathrm{e}^{-2x}\sin x+c_2\mathrm{e}^{-2x}\cos x$$

$$2=y'(0)=-2c_1\mathrm{e}^0\cos 0-c_1\mathrm{e}^0\sin 0-2c_2\mathrm{e}^0\sin 0+c_2\mathrm{e}^0\cos 0=-2c_1+c_2.$$

As $c_1=1$, thus $c_2=4$. The solution of the initial value problem:

$$y(x)=\mathrm{e}^{-2x}\cos x+4\mathrm{e}^{-2x}\sin x.$$

Exercise 3.4.4 *Solve the following differential equation:* $y''(x)+5y'(x)+6y(x)=12\mathrm{e}^x$.

Solution.

1. the auxiliary equation: $\lambda^2+5\lambda+6=0$, the roots: $\lambda_1=-2$, $\lambda_2=-3$. The general solution of the homogenization:

$$y_c(x)=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{-3x} \qquad (c_1,c_2\in\mathbb{R}).$$

2. we search a particular solution of the inhomogeneous DE in the form

$$y_p(x)=c_1(x)\mathrm{e}^{-2x}+c_2(x)\mathrm{e}^{-3x}$$

by solving the system

$$\begin{aligned} c_1'(x)\cdot\mathrm{e}^{-2x}+{}&c_2'(x)\cdot\mathrm{e}^{-3x}=0\\ c_1'(x)\cdot(-2)\cdot\mathrm{e}^{-2x}+{}&c_2'(x)\cdot(-3)\cdot\mathrm{e}^{-3x}=12\mathrm{e}^x\end{aligned}$$

This is a linear system of equations for $c_1'(x)$ and $c_2'(x)$, we solve it by Cramer's rule. The Wronskian (i.e. the coefficient matrix of the system):

$$W(x) = \begin{vmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-3x} \\ -2\mathrm{e}^{-2x} & -3\mathrm{e}^{-3x} \end{vmatrix} = -\mathrm{e}^{-5x}.$$

Denote $W_i(x)$ $(i = 1, 2)$ the determinant which one we get if we replaced the i-th coloumn of the Wronskian with the vector $\begin{pmatrix} 0 \\ r(x) \end{pmatrix}$. Then we have

$$W_1(x) = \begin{vmatrix} 0 & \mathrm{e}^{-3x} \\ 12\mathrm{e}^{x} & -3\mathrm{e}^{-3x} \end{vmatrix} = -12\mathrm{e}^{-2x}.$$

$$W_2(x) = \begin{vmatrix} \mathrm{e}^{-2x} & 0 \\ -2\mathrm{e}^{-2x} & 12\mathrm{e}^{x} \end{vmatrix} = 12\mathrm{e}^{-x}.$$

By Cramer' rule we get, that

$$c_1'(x) = \frac{W_1(x)}{W(x)} = \frac{-12\mathrm{e}^{-2x}}{-\mathrm{e}^{-5x}} = 12\mathrm{e}^{3x}, \quad c_2'(x) = \frac{W_2(x)}{W(x)} = \frac{12\mathrm{e}^{-x}}{-\mathrm{e}^{-5x}} = -12\mathrm{e}^{4x},$$

from which it follows that

$$c_1(x) = 12 \int \mathrm{e}^{3x}\,\mathrm{d}x = 12\frac{\mathrm{e}^{3x}}{3} = 4\mathrm{e}^{3x}$$

and

$$c_2(x) = -12 \int \mathrm{e}^{4x}\,\mathrm{d}x = -12\frac{\mathrm{e}^{4x}}{4} = -3\mathrm{e}^{4x}.$$

Using these functions

$$y_p(x) = 4\mathrm{e}^{3x}\mathrm{e}^{-2x} - 3\mathrm{e}^{4x}\mathrm{e}^{-3x} = \mathrm{e}^{x}.$$

3. $y(x) = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-3x} + \mathrm{e}^{x} \quad (c_1, c_2 \in \mathbb{R})$

Exercise 3.4.5 *Solve the following differential equation:* $y''(x) + y(x) = 3 \cdot \sin(2x)$.

Solution.

1. the auxiliary equation: $\lambda^2 + 1 = 0$, the roots: $\lambda_1 = -i$, $\lambda_2 = i$. The general solution of the homogenization:

$$y_c(x) = c_1 \cdot \cos x + c_2 \cdot \sin x \qquad (c_1, c_2 \in \mathbb{R}).$$

2. we search a particular solution of the inhomogeneous DE in the form

$$y_p(x) = c_1(x) \cdot \cos x + c_2(x) \cdot \sin x$$

by solving the system

$$\begin{aligned} c_1'(x) \cdot \cos x + \quad c_2'(x) \cdot \sin x &= 0 \\ c_1'(x) \cdot (-\sin x) + c_2'(x) \cdot \cos x &= 3 \cdot \sin(2x) \end{aligned}$$

This is a linear system of equations for $c_1'(x)$ and $c_2'(x)$, we solve it by Cramer's rule. The Wronskian (i.e. the coefficient matrix of the system):

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

Denote $W_i(x)$ $(i = 1, 2)$ the determinant which one we get if we replaced the i-th coloumn of the Wronskian with the vector $\begin{pmatrix} 0 \\ r(x) \end{pmatrix}$. Then we have

$$W_1(x) = \begin{vmatrix} 0 & \sin x \\ 3 \cdot \sin(2x) & \cos x \end{vmatrix} = -\sin x \cdot 3 \cdot \sin(2x) = -6 \cdot \sin^2 x \cdot \cos x.$$

$$W_2(x) = \begin{vmatrix} \cos x & 0 \\ -\sin x & 3 \cdot \sin(2x) \end{vmatrix} = \cos x \cdot 3 \cdot \sin(2x) = 6 \cdot \sin x \cdot \cos^2 x.$$

By Cramer' rule we get, that

$$c_1'(x) = \frac{W_1(x)}{W(x)} = -6 \cdot \sin^2 x \cdot \cos x, \quad c_2'(x) = \frac{W_2(x)}{W(x)} = 6 \cdot \sin x \cdot \cos^2 x,$$

from which (by $\int g' \cdot g^\alpha = \frac{g^{\alpha+1}}{\alpha+1}$ if $\alpha \neq -1$) it follows that

$$c_1(x) = -6 \cdot \frac{\sin^3 x}{3} = -2\sin^3 x$$

and

$$c_2(x) = -6 \cdot \frac{\cos^3 x}{3} = -2\cos^3 x.$$

Using these functions
$y_p(x) = -2\sin^3 x \cdot \cos x + (-2\cos^3 x \sin x) =$
$-2 \cdot \sin x \cdot \cos x \cdot (\sin^2 x + \cos^2 x) = -\sin(2x).$

3. $y(x) = c_1 \cdot \cos x + c_2 \cdot \sin x - \sin(2x) \quad (c_1, c_2 \in \mathbb{R})$

Exercise 3.4.6 *Solve the following DE:* $y''(x) + y'(x) - 2y(x) = 5e^{4x}$

Solution:

1. the complementary solution y_c of the homogeneous part:

$$\lambda^2 + \lambda - 2 = 0 \;\Rightarrow\; \lambda_1 = -2, \quad \lambda_2 = 1$$

$$y_c(x) = c_1 \cdot e^{-2x} + c_2 \cdot e^x$$

2. a particular solution y_p of the inhomogeneous part: $y_p(x) = A \cdot e^{4x}$

$$16Ae^{4x} + 4Ae^{4x} - 2Ae^{4x} = 5e^{4x} \;\Rightarrow\; 18A = 5 \;\Rightarrow\; A = \frac{5}{18}$$

$$y_p(x) = \frac{5}{18} \cdot e^{4x}$$

3. the general solution of the inhomogeneous DE:

$$y(x) = y_c(x) + y_p(x) = c_1 \cdot e^{-2x} + c_2 \cdot e^x + \frac{5}{18} \cdot e^{4x}$$

Exercise 3.4.7 *Solve the following DE:* $y''(x) + y(x) = 3\sin(2x)$

Solution:

1. the complementary solution y_c of the homogeneous part:

$$\lambda^2 + 1 = 0 \;\Rightarrow\; \lambda_1 = -i, \quad \lambda_2 = i$$

$$y_c(x) = c_1 \cdot e^{0 \cdot x} \cdot \cos(x) + c_2 \cdot e^{0 \cdot x} \cdot \sin(x) = c_1 \cos(x) + c_2 \sin(x)$$

2. a particular solution y_p of the inhomogeneous part:

$$y_p(x) = B\cos(2x) + C\sin(2x)$$

$$(B\cos(2x) + C\sin(2x))'' + B\cos(2x) + C\sin(2x) = 3\sin(2x)$$
$$(-2B\sin(2x) + 2C\cos(2x))' + B\cos(2x) + C\sin(2x) = 3\sin(2x)$$
$$-4B\cos(2x) - 4C\sin(2x) + B\cos(2x) + C\sin(2x) = 3\sin(2x)$$

$$-3B\cos(2x) - 3C\sin(2x) = 3\sin(2x)$$
$$\Rightarrow \quad B = 0, C = -1 \;\Rightarrow\; \quad y_p(x) = -\sin(2x)$$

3. the general solution of the inhomogeneous DE:

$$y(x) = c_1\cos(x) + c_2\sin(x) - \sin(2x)$$

Exercise 3.4.8 *Solve the following initial value problem:*

$$y''(x) + 2y'(x) + y(x) = x^2, \; y(0) = 7, \; y'(0) = -3$$

Solution:

1. the complementary solution y_c of the homogeneous part:

$$\lambda^2 + 2\lambda + 1 = 0 \;\Rightarrow\; \lambda = -1$$

$$y_c(x) = c_1 \cdot e^{-x} + c_2 \cdot x \cdot e^{-x}$$

2. a particular solution y_p of the inhomogeneous part:

$$y_p(x) = Ax^2 + Bx + C$$

$$(Ax^2 + Bx + C)'' + 2(Ax^2 + Bx + C)' + Ax^2 + Bx + C = x^2$$

$$2A + 4Ax + 2B + Ax^2 + Bx + C = x^2 \;\Rightarrow\; A = 1,\ B = -4,\ C = 6$$

$$y_p(x) = x^2 - 4x + 6$$

3. the general solution of the inhomogeneous DE:

$$y(x) = c_1 \cdot e^{-x} + c_2 \cdot x \cdot e^{-x} + x^2 - 4x + 6$$

4. $7 = y(0) = c_1 + 6$ and $-3 = y'(0) = -c_1 + c_2 - 4$, the solution of the system: $c_1 = 1$, $c_2 = 2$

   The solution of the initial value problem:

$$y(x) = e^{-x} + 2 \cdot x \cdot e^{-x} + x^2 - 4x + 6$$

3.5 Laplace transform and its applications

Definition 3.5.1 *Let $f : (0, \infty) \to \mathbb{R}$ be a function which is continuous up to finitely many points, and $f(t)e^{-at}$ is bounded for some positive real number a. Then the Laplace transform of f is defined as*

$$\mathcal{L}[f(t)] = F(s) = \int_a^\infty f(t)e^{-st}dt.$$

The inverse Laplace transform is given by

$$\mathcal{L}^{-1}[F(s)] = f(t).$$

The basic Laplace transforms are given in the following table.

Basic Laplace transforms

$f(t)$	1	$e^{-at}t^n$ $(a \geq 0, n \geq 0)$	$\sin \omega t$	$\cos \omega t$
$F(s) = \mathcal{L}[f(t)]$	$\frac{1}{s}$	$\frac{n!}{(s+a)^{n+1}}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2+\omega^2}$

Theorem 3.5.1 *The Laplace transform is linear, that is we have*

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

for any $a, b > 0$ and functions f, g as above.

Remark 3.5.1 *The Laplace transform is not multiplicative, that is, in general we have*

$$\mathcal{L}[f(t)g(t)] \neq \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

for functions f, g as above.

Theorem 3.5.2 *Let f be as above, and let $F(s) = \mathcal{L}[f(t)]$. Then we have*

$$\mathcal{L}[f'(t)] = sF(s) - f(0), \quad \mathcal{L}[e^{-at}f(t)] = F(s+a) \ (a \geq 0),$$

$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$ *and*

$$f(0) = \lim_{s\to\infty} sF(s).$$

Application of Laplace transform for differential equations

Let b_0, b_1, b_2 be real numbers, and consider the differential equation

$$b_2y'' + b_1y' + b_0y = f(t),$$

assuming that for all functions appearing here we can apply Laplace transform. Then we have

$$\mathcal{L}[b_2y'' + b_1y' + b_0y] = \mathcal{L}[f(t)],$$

yielding

$$b_2\mathcal{L}[y''] + b_1\mathcal{L}[y'] + b_0\mathcal{L}[y] = F(s),$$

where $F(s) = \mathcal{L}[f(t)]$. Letting $Y(s) = \mathcal{L}[y(t)]$, we get

$$b_2(s^2Y(s) - sy(0) - y'(0)) + b_1(sY(s) - y(0)) + b_0Y(s) = F(s).$$

This gives

$$Y(s) = \frac{F(s)}{b_2s^2 + b_1s + b_0} + \frac{sb_2y(0) + b_2y'(0) + b_1y(0)}{b_2s^2 + b_1s + b_0}.$$

From this, using inverse Laplace transform, we obtain y.

Exercise 3.5.1 *Calculate the Laplace transforms of the constant function 1.*

If $f(t) = 1$ then

$$\mathcal{L}[1] = F(s) = \int_0^\infty 1 \cdot e^{-st}\mathrm{d}\,t = \lim_{\varepsilon\to\infty}\int_0^\varepsilon 1 \cdot e^{-st}\mathrm{d}\,t =$$

$$= \lim_{\varepsilon\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^\varepsilon = \lim_{\varepsilon\to\infty}\left(\frac{e^{-s\varepsilon}}{-s} - \frac{1}{-s}\right) = \lim_{\varepsilon\to\infty}\left(\frac{1}{s} - \frac{e^{-s\varepsilon}}{s}\right) = \frac{1}{s}$$

Exercise 3.5.2 *Solve the following initial value problem:*

$$y'' - 6y' + 5y = 0,\ \ y(0) = 1,\ \ y'(0) = -3$$

Solution.

STEP1: Transform both sides:

$$\mathcal{L}[y'' - 6y' + 5y] = \mathcal{L}[0]$$

using the linearity of the transformation we have

$$\mathcal{L}[y''] - 6\mathcal{L}[y'] + 5\mathcal{L}[y] = 0.$$

Using Theorem $\star$ we get that

$$s^2\mathcal{L}[y] - sy(0) - y'(0) - 6(s\mathcal{L}[y] - y(0)) + 5\mathcal{L}[y] = 0$$

STEP2: Simplify to find $Y(s) = \mathcal{L}[y]$:

$$s^2\mathcal{L}[y] - sy(0) - (-3) - 6(s\mathcal{L}[y] - 1) + 5\mathcal{L}[y] = 0$$

$$(s^2 - 6s + 5)\mathcal{L}[y] - s + 9 = 0$$

$$\mathcal{L}[y] = \frac{s-9}{s^2 - 6s + 5}$$

STEP3: Find the inverse transform $y(t)$:
Use partial fractions to simplify, $(s^2 - 6s + 5 = 0 \;\Rightarrow\; s_1 = 1,\; s_2 = 5)$

$$\frac{s-9}{s^2 - 6s + 5} = \frac{a}{s-1} + \frac{b}{s-5}$$

$$s - 9 = a(s-5) + b(s-1) = (a+b)s + (-5a - b)$$

Equating the corresponding coefficients we get system of equations: $\{1 = a + b,\; -9 = -5a - b\}$, the solution is $a = 2$, $b = -1$. Thus $\mathcal{L}[y] = \frac{2}{s-1} - \frac{1}{s-5}$. The last expression corresponds to the Laplace transform of $2e^t - e^{5t}$. Therefore, it must be that

$$y(t) = 2e^t - e^{5t}$$

REMARK: The method finds the particular solution of an initial value problem directly, without solving for the general solution first.

Exercise 3.5.3 *Solve the following initial value problem:*

$$y'' - 2y' + 2y = \cos t, \quad y(0) = 1,\; y'(0) = 0$$

Solution. STEP1: Transform both sides:

$$s^2\mathcal{L}[y] - sy(0) - y'(0) - 2(s\mathcal{L}[y] - y(0)) + 2\mathcal{L}[y] = \mathcal{L}[\cos t]$$

STEP2: Simplify to find $Y(s) = \mathcal{L}[y]$:

$$(s^2 - 2s + 2)\mathcal{L}[y] - s + 2 = \frac{s}{s^2+1}$$

$$(s^2 - 2s + 2)\mathcal{L}[y] = \frac{s}{s^2+1} + s - 2$$

$$(s^2 - 2s + 2)\mathcal{L}[y] = \frac{s + (s-2)(s^2+1)}{s^2+1}$$

$$\mathcal{L}[y] = \frac{s + s^3 + s - 2s^2 - 2}{(s^2+1)(s^2-2s+2)}$$

$$\mathcal{L}[y] = \frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)}$$

STEP3: Find the inverse transform $y(t)$:
Use partial fractions to simplify.
$s^2 + 1 \neq 0,\ s^2 - 2s + 2 \neq 0$ if $s \in \mathbb{R}$:

$$\frac{s^3 - 2s^2 + 2s - 2}{(s^2+1)(s^2-2s+2)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2-2s+2}$$

If we multiply by $(s^2+1)(s^2-2s+2)$ then we have
The left-hand side: $s^3 - 2s^2 + 2s - 2$
The right-hand side:

$$(as+b)(s^2-2s+2) + (cs+d)(s^2+1) =$$

$$= as^3 - 2as^2 + 2as + bs^2 - 2bs + 2b + cs^3 + cs + ds^2 + d =$$

$$= (a+c)s^3 + (b+d-2a)s^2 + (2a-2b+c)s + 2b + d$$

Equating the corresponding coefficients we get system of equations:

$$\{a + c = 1,\ b + d - 2a = -2,\ 2a - 2b + c = 2,\ 2b + d = -2\}$$

Now we solve this linear system of equation by Gauss elimination: The fourth row is a short notation for the equation $5d = -6$, thus $d = -\frac{6}{5}$
From the third row $12c + 8 \cdot \left(-\frac{6}{5}\right) = 0$, thus $c = \frac{4}{5}$
From the second equation $b + 2 \cdot \frac{4}{5} + \left(-\frac{6}{5}\right) = 0$, thus $b = -\frac{2}{5}$

1	0	1	0	1	
−2	1	0	1	−2	$+2 \cdot Row1$
2	−2	1	0	2	$-2 \cdot Row1$
0	2	0	1	−2	
1	0	1	0	1	
0	1	2	1	0	
0	−2	−1	0	0	$+2 \cdot Row2$
0	2	0	1	−2	$-2 \cdot Row2$
1	0	1	0	1	
0	1	2	1	0	
0	0	3	2	0	$\cdot 4$
0	0	−4	−1	−2	$\cdot 3$
1	0	1	0	1	
0	1	2	1	0	
0	0	12	8	0	
0	0	−12	−3	−6	$+Row3$
1	0	1	0	1	
0	1	2	1	0	
0	0	12	8	0	
0	0	0	5	−6	

From the first equation $a + \frac{4}{5} = 1$, thus $a = \frac{1}{5}$

$$\mathcal{L}[y] = \frac{\frac{1}{5}s - \frac{2}{5}}{s^2+1} + \frac{\frac{4}{5}s - \frac{6}{5}}{s^2-2s+2}$$

$$\mathcal{L}[y] = \frac{1}{5} \cdot \left(\frac{s-2}{s^2+1} + \frac{4s-6}{s^2-2s+2} \right)$$

$$\mathcal{L}[y] = \frac{1}{5} \cdot \left(\frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{4(s-1)}{s^2-2s+2} - \frac{2}{s^2-2s+2} \right)$$

$$\mathcal{L}[y] = \frac{1}{5} \cdot \left(\frac{s}{s^2+1} - 2\frac{1}{s^2+1} + 4\frac{s-1}{(s-1)^2+1} - 2\frac{1}{(s-1)^2+1} \right)$$

$$y(t) = \frac{1}{5} \left(\cos t - 2\sin t + 4e^t \cos t - 2e^t \sin t \right)$$

3.6 Special second order differential equations

Consider the differential equation $y''(x) = f(x, y(x), y'(x))$.

- If $f(x, y'(x))$ (it is independent from y), then we can use substitution $y'(x) = p(x)$ to get an first order DE for p. If $p^\star$ denotes its solution, then the solution of the original second order DE is $\int p^\star$.

- If $f(y(x), y'(x))$ (it is independent from x), then we can use substitution $y' = p(y)$. In this case $y'' = p'(y) \cdot y' = p'(y) \cdot p(y)$. thus for p equality $p'(y)p(y) = f(y, p(y))$ holds. If $p^\star$ denotes its solution, to get the solution of the original second order DE we have to solve the separable equation $y' = p^\star(y)$.

- there is no general method if f is independent from y'.

Exercise 3.6.1 *Solve the following initial value problem*

$$y''(x) = 6x + \cos x, \qquad y(0) = 1,\ y'(0) = 2$$

After the substitution we have $p'(x) = 6x + \cos x$. By integration we get its solution:

$$p(x) = \int 6x + \cos x \,\mathrm{d}x = 3x^2 + \sin x + c$$

Because $p(x) = y'(x)$ we have $y'(x) = 3x^2 + \sin x + c$. By integration

the general solution: $y(x) = x^3 - \cos x + cx + d \quad (c, d \in \mathbb{R})$.

$$y(0) = 1 \ \text{ thus } \ 1 = -\cos 0 + d = -1 + d \ \Rightarrow d = 2;$$
$$y'(0) = 2 \ \text{ thus } \ 2 = 3 \cdot 0^2 + \sin 0 + c \ \Rightarrow c = 2.$$

The solution of the initial value problem:

$$y(x) = x^3 - \cos x + 2x + 2.$$

Exercise 3.6.2 *Solve the following initial value problem*

$$x \cdot y''(x) = y'(x), \qquad y(1) = 0,\, y'(1) = 2$$

Solution. $x \cdot p'(x) = p(x) \ \Rightarrow \ p'(x) - \frac{1}{x} \cdot p(x) = 0$ - first order linear homogeneous DE

$$\Rightarrow p(x) = C \cdot e^{-\int -\frac{1}{x}\mathrm{d}x} = C \cdot e^{\ln|x|} = C \cdot |x|$$

As $y'(x) = p(x)$, thus $y'(x) = C \cdot |x|$ - By integration the general solution: $y(x) = C\frac{x^2}{2} + D \ \ (C, D \in \mathbb{R})$.

$$y'(x) = C \cdot x \quad \Rightarrow \quad 2 = y'(1) = C \cdot 1 \quad \Rightarrow \quad C = 2$$

$$0 = y(1) = \frac{C}{2} + D = 1 + D \quad \Rightarrow \quad D = -1$$

The solution of the initial value problem:

$$y(x) = x^2 - 1.$$

Exercise 3.6.3 *Solve the following DE* $y''(x) + 3y'(x) = x$

Solution.

$$p'(x) + 3p(x) = x \ \text{ - first order linear inhomogeneous DE}$$

The homogenization: $p'(x) + 3p(x) = 0 \ \Rightarrow \ p_h(x) = C \cdot e^{-\int 3\mathrm{d}x} = C \cdot e^{-3x}$.
A particular solution of the inhomogeneous equation: $p_p(x) = C(x) \cdot \mathrm{e}^{-3x}$

$$\left(C(x) \cdot \mathrm{e}^{-3x}\right)' + 3 \cdot C(x) \cdot \mathrm{e}^{-3x} = x$$

$$C'(x) \cdot \mathrm{e}^{-3x} - 3 \cdot C(x) \cdot \mathrm{e}^{-3x} + 3 \cdot C(x) \cdot \mathrm{e}^{-3x} = x,$$

i.e. $C'(x) = x\mathrm{e}^{3x}$.

Now we use the partial integration rule to get $C(x)$:

$$C(x) = \frac{1}{3}x\mathrm{e}^{3x} - \int \frac{1}{3}\mathrm{e}^{3x}\,\mathrm{d}x = \frac{1}{3}x\mathrm{e}^{3x} - \frac{1}{9}\mathrm{e}^{3x} = \mathrm{e}^{3x}\left(\frac{1}{3}x - \frac{1}{9}\right).$$

Using this we get that

$$p_p(x) = C(x)\mathrm{e}^{-3x} = \mathrm{e}^{3x}\left(\frac{1}{3}x - \frac{1}{9}\right)\mathrm{e}^{-3x} = \frac{1}{3}x - \frac{1}{9}.$$

The general solution of the first order inhomogeneous DE:

$$p(x) = p_h(x) + p_p(x) = C \cdot \mathrm{e}^{-3x} + \frac{1}{3}x - \frac{1}{9}.$$

As $y'(x) = p(x)$, thus

$$y'(x) = C \cdot \mathrm{e}^{-3x} + \frac{1}{3}x - \frac{1}{9} \ \text{, by integration we get that}$$

$$y(x) = C \cdot \frac{\mathrm{e}^{-3x}}{-3} + \frac{x^2}{6} - \frac{x}{9} + D \qquad (C, D \in \mathbb{R}).$$

Exercise 3.6.4 *Solve the following DE:*

$$y''(x) = 2 \cdot y'(x) \cdot y(x)$$

Solution. After the substitution we have $p'(y) \cdot p(y) = 2 \cdot p(y) \cdot y$. It is easy to see that $p = 0$ is a solution. Assume that $p \neq 0$. Then $p'(y) = 2 \cdot y$, therefore by integration we get that $p(y) = y^2 + c$. As $y' = p(y)$, thus $y' = y^2 + c \quad \Rightarrow \; y' = 1 \cdot (y^2 + c)$, which is a separable equation.
With the notations $h(y) = y^2 + c, \; g(x) = 1$ using the solution method $\int \frac{1}{h(y)} \, \mathrm{d}y = \int g(x) \, \mathrm{d}x$ we have

$$\int \frac{1}{y^2 + c} \, \mathrm{d}y = \int 1 \, \mathrm{d}x.$$

$\int 1 \, \mathrm{d}x = x + d_1$,
on the other hand

$$\int \frac{1}{y^2 + c} \, \mathrm{d}y = \frac{1}{c} \int \frac{1}{\frac{y^2}{c} + 1} \, \mathrm{d}y = \frac{1}{c} \int \frac{1}{\left(\frac{y}{\sqrt{c}}\right)^2 + 1} \, \mathrm{d}y =$$

$$= \frac{1}{c} \cdot \sqrt{c} \int \frac{1}{\left(\frac{y}{\sqrt{c}}\right)^2 + 1} \cdot \frac{1}{\sqrt{c}} \, \mathrm{d}y \overbrace{=}^{\int f(g) \cdot g' = F(g) + constant}$$

$$= \frac{1}{c} \cdot \sqrt{c} \cdot \operatorname{arctg}\left(\frac{y}{\sqrt{c}}\right) + d_2 = \frac{1}{\sqrt{c}} \operatorname{arctg}\left(\frac{y}{\sqrt{c}}\right) + d_2,$$

using the notation $d := d_1 - d_2$ we have

$$\frac{1}{\sqrt{c}} \operatorname{arctg}\left(\frac{y}{\sqrt{c}}\right) = x + d.$$

$$y(x) = \sqrt{c} \operatorname{tg}(\sqrt{c}x + \sqrt{c}d) = C \operatorname{tg}(Cx + D).$$

Exercise 3.6.5 *Solve the following DE:*

$$y(x) \cdot y''(x) = \left(y'(x)\right)^2$$

After the substitution we have $y \cdot p'(y) \cdot p(y) = (p(y))^2$. It is easy to see that $p = 0$ is a solution. Assume that $p \neq 0$. Then we get $y \cdot p'(y) = p(y)$ which is an first order linear homogeneous DE. It can be written in the form $p'(y) - \frac{1}{y} \cdot p(y) = 0$. Thus

$$p(y) = C \cdot e^{-\int -\frac{1}{y} \mathrm{d}y} = C \cdot e^{ln|y|} = C \cdot |y|$$

As $y' = p(y)$, thus $y' = C \cdot y$ which is an first order linear homogeneous DE. It can be written in the form $y' - C \cdot y = 0$

$$y(x) = D \cdot e^{-\int -C \mathrm{d}x} = D \cdot e^{C \cdot x}$$

Exercise 3.6.6 *Solve the following initial value problem*

$$y''(x) = 6x + \cos x, \qquad y(0) = 1,\, y'(0) = 2$$

Solution Using substitution $p(x) = y'(x)$ we have that $p'(x) = y''(x)$, thus

$$p'(x) = 6x + \cos x.$$

By integration we get that

$$p(x) = \int 6x + \cos x \,\mathrm{d}x = 3x^2 + \sin x + c.$$

As $p(x) = y'(x)$, therefore

$$y'(x) = 3x^2 + \sin x + c.$$

Using integration again

$$y(x) = x^3 - \cos x + cx + d \quad (c, d \in \mathbb{R}).$$

From the initial assumption $y(0) = 1$, it follows that

$$1 = -\cos 0 + d = -1 + d$$

thus $d = 2$.

On the other hand $y'(0) = 2$, thus $2 = 3 \cdot 0^2 + \sin 0 + c$, i.e. $c = 2$. The solution of the initial value problem:

$$y(x) = x^3 - \cos x + 2x + 2.$$

4

Differentiation for functions of type $\mathbb{R}^n \to \mathbb{R}^m$

4.1 Scalar fields

Exercise 4.1.1 *Give the first order partial derivatives of*

$$f(x,y) = x^2 + 3xy + y^3$$

Solution. The notation of the first order partial derivatives:

$$f'_x(x,y) = \partial_x f(x,y) = \partial_1 f(x,y) = D_1 f(x,y) = \frac{\partial f}{\partial x}(x,y)$$
$$f'_y(x,y) = \partial_y f(x,y) = \partial_2 f(x,y) = D_2 f(x,y) = \frac{\partial f}{\partial y}(x,y)$$

For the determination of $\partial_1 f(x,y)$ we have to think of x as a variable and y as a constant. Similarly, for $\partial_2 f(x,y)$ we have to think of y as a variable and x as a constant.

$$\partial_1 f(x,y) = 2x + 3y \quad \text{and} \quad \partial_2 f(x,y) = 3x + 3y^2$$

Exercise 4.1.2 *Give the first order partial derivatives of*

$$f(x,y) = x^2 \sin(x^3 + y^4)$$

Solution.

$$\frac{\partial f}{\partial x}(x,y) = 2x\sin(x^3+y^4) + x^2\big(\cos(x^3+y^4)\big)\cdot 3x^2 =$$

$$= 2x\sin(x^3+y^4) + 3x^4\cos(x^3+y^4)$$

$$\frac{\partial f}{\partial y}(x,y) = 4x^2y^3\cos(x^3+y^4).$$

Exercise 4.1.3 *Give the first order partial derivatives of*

$$f(x,y) = \frac{x^5+xy}{(x+2y)^3}$$

Solution.

$$\frac{\partial f}{\partial x}(x,y) = \frac{(5x^4+y)(x+2y)^3 - (x^5+xy)\cdot 3\cdot(x+2y)^2\cdot 1}{(x+2y)^6}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x(x+2y)^3 - (x^5+xy)\cdot 3\cdot(x+2y)^2\cdot 2}{(x+2y)^6}$$

Exercise 4.1.4 *Give the first order partial derivatives of*

$$f(x,y) = x^5\cdot \mathrm{e}^{2xy}$$

Solution.

$$\frac{\partial f}{\partial x}(x,y) = 5x^4\cdot \mathrm{e}^{2xy} + x^5\mathrm{e}^{2xy}\cdot 2y$$

$$\frac{\partial f}{\partial y}(x,y) = x^5\mathrm{e}^{2xy}\cdot 2x = 2x^6\mathrm{e}^{2xy}$$

Exercise 4.1.5 *Give the first order partial derivatives of*

$$f(x,y) = x^y + y^x$$

Solution.

$$\frac{\partial f}{\partial x}(x,y) = yx^{y-1} + y^x \cdot \ln y$$

$$\frac{\partial f}{\partial y}(x,y) = x^y \cdot \ln x + xy^{x-1}$$

Exercise 4.1.6 *Give the first order partial derivatives of*

$$f(x,y) = \sqrt{x^2 + 6xy + y^3}$$

Solution. The root can be written as a power:

$$f(x,y) = (x^2 + 6xy + y^3)^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2}(x^2 + 6xy + y^3)^{-\frac{1}{2}}(2x + 6y)$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{2}(x^2 + 6xy + y^3)^{-\frac{1}{2}}(6x + 3y^2)$$

Remark 4.1.1 *The derivative of scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ at $\underline{r}_0$ is called the gradient of f at $\underline{r}_0$ and it can be calculated as*

$$\operatorname{grad} f(\underline{r}_0) = (\partial_1 f(\underline{r}_0), \ldots, \partial_n f(\underline{r}_0)) \,.$$

Exercise 4.1.7 *Give the value of the derivative of the following function at the given place.* $f(x,y,z) = (2x-z)\cdot(2-y)^3$, $\underline{r}_0 = (x_0, y_0, z_0) = (1,0,0)$

Solution.

$$\partial_1 f(x,y,z) = 2\cdot(2-y)^3 + (2x-z)\cdot 0 = 2\cdot(2-y)^3$$

$$\partial_2 f(x,y,z) = (2x-z)\cdot 3\cdot(2-y)^2\cdot(-1)$$

$$\partial_3 f(x,y,z) = (-1)\cdot(2-y)^3$$

$\operatorname{grad} f(x,y,z) = \big(2\cdot(2-y)^3, -3\cdot(2x-z)\cdot(2-y)^2, (-1)\cdot(2-y)^3\big)_{1\times 3}$, thus $\operatorname{grad} f(1,0,0) = (16,-24,-8)$

Exercise 4.1.8 *Give the gradients of the scalar fields at $\underline{r}_0$.*

- $f(x,y) = x^2+5xy+y$, $\underline{r}_0 = (2,-1)$
- $f(x,y) = \frac{x^2}{16}+\frac{y^2}{12}-1$, $\underline{r}_0 = (2,-3)$
- $f(x,y,z) = x^3+5xy^2+y+z^2$, $\underline{r}_0 = (2,-1,0)$

Solution.

- $\frac{\partial f}{\partial x}(x,y) = 2x+5y \quad\Rightarrow \frac{\partial f}{\partial x}(\underline{r}_0) = 4-5=-1$,
  $\frac{\partial f}{\partial y}(x,y) = 5x+1 \;\Rightarrow \frac{\partial f}{\partial y}(\underline{r}_0) = 10+1=11$.
  $\operatorname{grad} f(\underline{r}_0) = (-1,11)$.
- $\frac{\partial f}{\partial x}(x,y) = \frac{1}{8}x \quad\Rightarrow \frac{\partial f}{\partial x}(\underline{r}_0) = \frac{1}{4}$, $\frac{\partial f}{\partial y}(x,y) = \frac{1}{6}y \;\Rightarrow \frac{\partial f}{\partial y}(\underline{r}_0) = -\frac{1}{2}$.
  $\operatorname{grad} f(\underline{r}_0) = \left(\frac{1}{4},-\frac{1}{2}\right)$.
- $\frac{\partial f}{\partial x}(x,y,z) = 3x^2+5y^2 \;\Rightarrow \frac{\partial f}{\partial x}(\underline{r}_0) = 17$,
  $\frac{\partial f}{\partial y}(x,y,z) = 10xy+1 \;\Rightarrow \frac{\partial f}{\partial y}(\underline{r}_0) = -19$, $\frac{\partial f}{\partial z}(x,y,z) = 2z \;\Rightarrow \frac{\partial f}{\partial z}(\underline{r}_0) = 0$.
  $\operatorname{grad} f(\underline{r}_0) = (17,-19,0)$.

Remark 4.1.2 *The directional derivative of $f\colon D\ (\subset \mathbb{R}^n) \to \mathbb{R}^m$ at the point $\underline{r}_0$, along the direction $\underline{v}$ is the limit*

$$\partial_{\underline{v}} f(\underline{r}_0)\colon = \lim_{\lambda\to 0} \frac{f(\underline{r}_0 + \lambda \cdot \underline{v}^\circ) - f(\underline{r}_0)}{\lambda},$$

whenever it exists. Here $\underline{v}^\circ$ denote the unit vector which has the same direction as $\underline{v}$, i.e. $v^\circ = \frac{1}{||\underline{v}||} \cdot \underline{v}$.

It can be proved that the directional derivative of f can be calculated as

$$\partial_{\underline{v}} f(\underline{r}_0) = \operatorname{grad} f(\underline{r}_0) \bullet \underline{v}^\circ.$$

Exercise 4.1.9 *Give the directional derivative of f along the direction $\underline{v}$ at the point $\underline{r}_0$.*

a) $f(x,y) = x^2 + y^2$, $\underline{r}_0 = (1,3)$, $\underline{v} = (3,4)$

b) $f(x,y,z) = 2xy - yz$, $\underline{r}_0 = (1,-1,1)$, $\underline{v} = (1,3,-2)$

Solution. *a*) First solution method:

$$\underline{v}^\circ = \frac{1}{||\underline{v}||} \cdot \underline{v} = \frac{1}{\sqrt{3^2+4^2}} \cdot (3,4) = \frac{1}{5} \cdot (3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\partial_{\underline{v}} f(1,3)\colon = \lim_{\lambda\to 0} \frac{f\left((1,3) + \lambda \cdot \left(\frac{3}{5}, \frac{4}{5}\right)\right) - f(1,3)}{\lambda} =$$

$$= \lim_{\lambda\to 0} \frac{f\left(1 + \frac{3}{5}\lambda, 3 + \frac{4}{5}\lambda\right) - f(1,3)}{\lambda} =$$

$$= \lim_{\lambda\to 0} \frac{\left(1 + \frac{3}{5}\lambda\right)^2 + \left(3 + \frac{4}{5}\lambda\right)^2 - (1^2 + 3^2)}{\lambda} = \lim_{\lambda\to 0} \frac{(6+\lambda)\cdot\lambda}{\lambda} = \underline{\underline{6}}$$

Second solution method:

$$\underline{v}^\circ = \frac{1}{||\underline{v}||} \cdot \underline{v} = \frac{1}{\sqrt{3^2+4^2}} \cdot (3,4) = \frac{1}{5} \cdot (3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\frac{\partial f}{\partial x}(x,y) = 2x \qquad \frac{\partial f}{\partial y}(x,y) = 2y$$

$$\frac{\partial f}{\partial x}(\underline{r}_0) = 2\cdot 1 = 2 \qquad \frac{\partial f}{\partial y}(\underline{r}_0) = 2\cdot 3 = 6$$

$$\partial_{\underline{v}} f(1,3) = (2,6) \bullet \left(\frac{3}{5}, \frac{4}{5}\right) = 2\cdot\frac{3}{5} + 6\cdot\frac{4}{5} = \underline{\underline{6}}$$

b) $\underline{v}^\circ = \frac{1}{||\underline{v}||}\cdot \underline{v} = \frac{1}{\sqrt{1^2+3^2+(-2)^2}}\cdot(1,3,-2) = \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\right)$
$\frac{\partial f}{\partial x}(x,y,z) = 2y \quad \frac{\partial f}{\partial y}(x,y,z) = 2x - z \quad \frac{\partial f}{\partial z}(x,y,z) = -y$
$\operatorname{grad} f(x,y,z) = (2y, 2x-z, -y) \;\Rightarrow\; \operatorname{grad} f(\underline{r}_0) = (-2,1,1)$

$$\partial_{\underline{v}} f(1,-1,1) = (-2,1,1) \bullet \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\right) =$$

$$= (-2)\cdot\frac{1}{\sqrt{14}} + 1\cdot\frac{3}{\sqrt{14}} + 1\cdot\left(-\frac{2}{\sqrt{14}}\right) = \underline{\underline{-\frac{1}{\sqrt{14}}}}$$

Exercise 4.1.10 *The temperature of a body at the point* $\underline{r} = (x,y,z)$ *given by scalar field*

$$T(x,y,z) = x^3 + 5xy^2 + y + z^2.$$

Give the direction of the fastest decline in view of $\underline{r}_0 = (2,-1,0)$.

Solution.

$$\begin{aligned}
\frac{\partial T}{\partial x}(x,y,z) &= 3x^2 + 5y^2 &\Rightarrow\quad \frac{\partial T}{\partial x}(\underline{r}_0) &= 17\\
\frac{\partial T}{\partial y}(x,y,z) &= 10xy + 1 &\Rightarrow\quad \frac{\partial T}{\partial y}(\underline{r}_0) &= -19\\
\frac{\partial T}{\partial z}(x,y,z) &= 2z &\Rightarrow\quad \frac{\partial T}{\partial z}(\underline{r}_0) &= 0.
\end{aligned}$$

The gradient: $\operatorname{grad} T(\underline{r}_0) = (17,-19,0)$. The direction of the fastest decline: $\underline{v} = (-17,19,0)$.

Remark 4.1.3 *By the formula of linear approximation we now that if we have a differentiable function* $f\colon D\subset\mathbb{R}^2\to\mathbb{R}$*, then for a small neighbourhood of* $\underline{r}_0$ *there is the approximation* $f(\underline{r})-f(\underline{r}_0)\approx grad\ f(\underline{r}_0)\bullet(\underline{r}-\underline{r}_0)$*. With the notations* $\underline{r}_0=(x_0,y_0)$*,* $\underline{r}=(x,y)$ *the graphical image of scalar field*

$$S(x,y)=f(\underline{r}_0)+f'_x(\underline{r}_0)\cdot(x-x_0)+f'_y(\underline{r}_0)\cdot(y-y_0)$$

is called the tangent plane of f *at* $\underline{r}_0$*.*

So we have that for a small neighbourhood of $\underline{r}_0$ *there is the approximation* $f(\underline{r})\approx S(\underline{r})$*.*

Exercise 4.1.11 *Give the tangent plane of* f *at* $\underline{r}_0$ *if* $f(x,y)=\sin x\cdot\operatorname{tg} y$ *and* $\underline{r}_0=\left(\frac{\pi}{6},\frac{\pi}{4}\right)$*. By the help of the tangent plane give an approximation for the value of* $\sin 29^\circ\cdot\operatorname{tg}46^\circ$*.*

Solution.

$$\frac{\pi}{6}(rad)=30^\circ,\quad \frac{\pi}{4}(rad)=45^\circ,\quad f(\underline{r}_0)=\sin\frac{\pi}{6}\cdot\operatorname{tg}\frac{\pi}{4}=\frac{1}{2}$$

$$f'_x(x,y)=\tan y\cdot\cos x\quad\Rightarrow f'_x(\underline{r}_0)=\frac{\sqrt{3}}{2}$$

$$f'_y(x,y)=\sin x\cdot\frac{1}{\cos^2 y}\quad\Rightarrow f'_y(\underline{r}_0)=\frac{1}{2}\cdot\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}=1$$

The tangent plane:

$$S(x,y)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)+1\cdot\left(y-\frac{\pi}{4}\right)$$

$$29^\circ=\frac{29\pi}{180}(rad),\quad 46^\circ=\frac{46\pi}{180}(rad)$$

$$\sin 29^\circ\cdot\tan 46^\circ=\sin\frac{29\pi}{180}\cdot\tan\frac{46\pi}{180}=f\left(\frac{29\pi}{180},\frac{46\pi}{180}\right)=?$$

$$f\left(\frac{29\pi}{180},\frac{46\pi}{180}\right)\approx S\left(\frac{29\pi}{180},\frac{46\pi}{180}\right)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{29\pi}{180}-\frac{\pi}{6}\right)+1\cdot\left(\frac{46\pi}{180}-\frac{\pi}{4}\right)=$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{-\pi}{180}\right) + 1 \cdot \left(\frac{\pi}{180}\right) \approx$$

$$\approx 0,5 + 0,8660 \cdot (-0,0175) + 0.0175 = 0.5023$$

Exercise 4.1.12 *A company produces cylindrical containers. The expected sizes are as follows: the radius of the circle is 1.5 [m] and the height of the cylinder is 7.5 [m]. How sensitive is volume to small changes in radius and height?*

Solution. The volume of a cylinder depending on the radius and height in the following way:

$$V(R, m) = R^2 \cdot \pi \cdot m$$

To give the sensitivity we use linear approximation: Using notations $\Delta\underline{r} = \underline{r} - \underline{r}_0$ and $\Delta V = V(\underline{r}) - V(\underline{r}_0)$, if $\underline{r}_0 = (1.5, 7.5)$ then

$$\Delta V \approx grad\ V(\underline{r}_0) \bullet \Delta\underline{r}.$$

$$\Delta V \approx V'_R(\underline{r}_0) \cdot \Delta R + V'_m(\underline{r}_0) \cdot \Delta m.$$

$$V'_R(R, m) = 2 \cdot R \cdot \pi \cdot m \;\Rightarrow\; V'_R(\underline{r}_0) = V'_R(1.5, 7.5) = 22.5\pi$$

$$V'_m(R, m) = R^2 \cdot \pi \;\Rightarrow\; V'_m(\underline{r}_0) = V'_m(1.5, 7.5) = 2.25\pi$$

$$\Delta V \approx 22.5\pi \cdot \Delta R + 2.25\pi \cdot \Delta m$$

This means, that a unit change in radius results in a change of 22.5π units in volume, while a unit change in height results in a change of 2.25π units in volume. So The change in volume is 10 times as sensitive to the small change in radius as to its height.

Remark 4.1.4 *Suppose that the scalar field $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at the interior point $\underline{r}_0 \in D$. If $\partial_1 f(\underline{r}_0) = \partial_2 f(\underline{r}_0) = \ldots = \partial_n f(\underline{r}_0) = 0$ then $\underline{r}_0$ is called a stationary point. ($\underline{r}_0$ is called saddle point if no extreme value is attained at $\underline{r}_0$.)*

Suppose that the interior point $\underline{r}_0 \in D$ is a stationary point. With the so-called leading principal minors

$$\Delta_1(\underline{r}_0) := \partial_1\partial_1 f(\underline{r}_0) \quad \Delta_2(\underline{r}_0) := det \begin{pmatrix} \partial_1\partial_1 f(\underline{r}_0) & \partial_2\partial_1 f(\underline{r}_0) \\ \partial_1\partial_2 f(\underline{r}_0) & \partial_2\partial_2 f(\underline{r}_0) \end{pmatrix}$$

...

$$\Delta_n(\underline{r}_0) := det \begin{pmatrix} \partial_1\partial_1 f(\underline{r}_0) & \partial_2\partial_1 f(\underline{r}_0) & \dots & \partial_n\partial_1 f(\underline{r}_0) \\ \vdots & \vdots & & \vdots \\ \partial_1\partial_n f(\underline{r}_0) & \partial_2\partial_n f(\underline{r}_0) & \dots & \partial_n\partial_n f(\underline{r}_0) \end{pmatrix}$$

we can state that

A) if $\Delta_1(\underline{r}_0) > 0, \Delta_2(\underline{r}_0) > 0, \dots, \Delta_n(\underline{r}_0) > 0$, then f has a **strict local minimum** *at $\underline{r}_0$.*

B) if $\Delta_k(\underline{r}_0) < 0$, when k is odd and $\Delta_k(\underline{r}_0) > 0$, when k is even, then f has a **strict local maximum** *at $\underline{r}_0$.*

C) if B) doesn't hold and one of the leading principal minor is negative, then $\underline{r}_0$ is a **saddle point** *of f.*

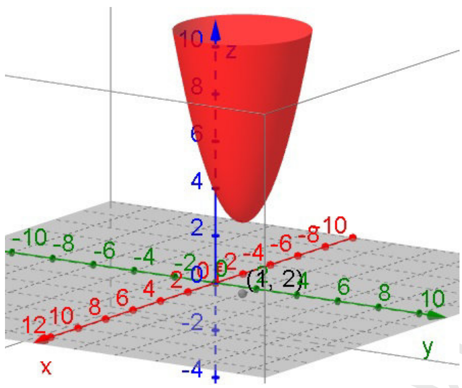
Exercise 4.1.13 *Give the local extremas of the following function:* $f(x,y) = (x-1)^2 + (y-2)^2 + 3 \quad (x,y) \in \mathbb{R}^2$

Solution.

- **step 1:** $\partial_1 f(x,y) = 2x - 2, \quad \partial_2 f(x,y) = 2y - 4$
- **step 2:** $2x - 2 = 0$ and $2y - 4 = 0 \Rightarrow \underline{r}_1 = (1,2)$
- **step 3:**

$$\partial_1\partial_1 f(x,y) = 2 \quad \partial_1\partial_2 f(x,y) = 0$$
$$\partial_2\partial_1 f(x,y) = 0 \quad \partial_2\partial_2 f(x,y) = 2$$

- **step 4:** $\partial_1\partial_1 f(\underline{r}_1) = 2,\ \partial_1\partial_2 f(\underline{r}_1) = 0,\ \partial_2\partial_1 f(\underline{r}_1) = 0,\ \partial_2\partial_2 f(\underline{r}_1) = 2$



- **step 5:** $\Delta_1(\underline{r}_1) = \partial_1\partial_1 f(\underline{r}_1) = 2 > 0$ and $\Delta_2(\underline{r}_1) = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 > 0 \Rightarrow$ positive definite
- **step 6:** f has a local minimum at $\underline{r}_0 = (1,2)$. The value of the minimum: $f(\underline{r}_0) = 3$

Remark 4.1.5 $\underline{r}_0 = (1,2)$ *is the absolute minimum of* f*. If* $x, y \in \mathbb{R}$*, then* $f(x,y) = (x-1)^2 + (y-2)^2 + 3 \geq 3$.

Exercise 4.1.14 *Give the local extremas of the following function:*

$$f(x,y) = -3x^3 - 3y^3 + x + 4y + 1$$

Solution. step 1:

$$\partial_1 f(x,y) = -9x^2 + 1 \quad \partial_2 f(x,y) = -9y^2 + 4$$

step 2:

$$I. \; -9x^2 + 1 = 0$$

$$II. \; -9y^2 + 4 = 0$$

From I. we get that $x_{1,2} = \pm\frac{1}{3}$, from II. we get that $y_{1,2} = \pm\frac{2}{3}$; the stationary points:

$$\underline{r}_1 = \left(\frac{1}{3}, \frac{2}{3}\right), \; \underline{r}_2 = \left(-\frac{1}{3}, \frac{2}{3}\right), \; \underline{r}_3 = \left(\frac{1}{3}, -\frac{2}{3}\right), \; \underline{r}_4 = \left(-\frac{1}{3}, -\frac{2}{3}\right)$$

step 3:

$$f''(x,y) = \begin{pmatrix} -18x & 0 \\ 0 & -18y \end{pmatrix}$$

step 4:

$$\Delta_2(\underline{r}_1) = \begin{vmatrix} -6 & 0 \\ 0 & -12 \end{vmatrix} \quad \Delta_2(\underline{r}_2) = \begin{vmatrix} 6 & 0 \\ 0 & -12 \end{vmatrix} \quad \Delta_2(\underline{r}_3) = \begin{vmatrix} -6 & 0 \\ 0 & 12 \end{vmatrix} \quad \Delta_2(\underline{r}_4) = \begin{vmatrix} 6 & 0 \\ 0 & 12 \end{vmatrix}$$

step 5:

- $\Delta_1(\underline{r}_1) = -6 < 0, \quad \Delta_2(\underline{r}_1) = 72 > 0 \quad \Rightarrow$ B)
- $\Delta_1(\underline{r}_2) = 6 > 0, \quad \Delta_2(\underline{r}_2) = -72 < 0 \quad \Rightarrow$ C)
- $\Delta_1(\underline{r}_3) = -6 < 0, \quad \Delta_2(\underline{r}_3) = -72 < 0 \quad \Rightarrow$ C)
- $\Delta_1(\underline{r}_4) = 6 > 0, \quad \Delta_2(\underline{r}_4) = 72 > 0 \quad \Rightarrow$ A)

step 6: f has a local maximum at $\underline{r}_1$, f has a local minimum at $\underline{r}_4$, $\underline{r}_2$ and $\underline{r}_3$ are saddle points. $\underline{r}_2 = \left(-\frac{1}{3}, \frac{2}{3}\right)$

Exercise 4.1.15 *Give the local extremas of the following function:*

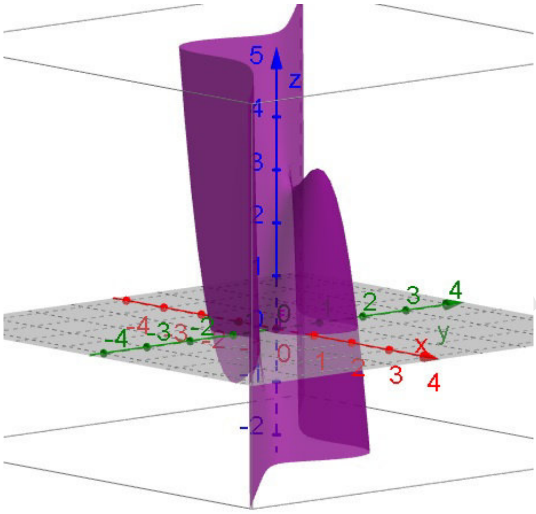
$$f(x,y,z) = x^3 + y^2 + z^2 + 12xy + 2z$$

Solution. step 1:

$$\partial_1 f(x,y,z) = 3x^2 + 12y \quad \partial_2 f(x,y,z) = 2y + 12x \quad \partial_3 f(x,y,z) = 2z + 2$$

step 2:

$$I.\; 3x^2 + 12y = 0$$
$$II.\; 2y + 12x = 0$$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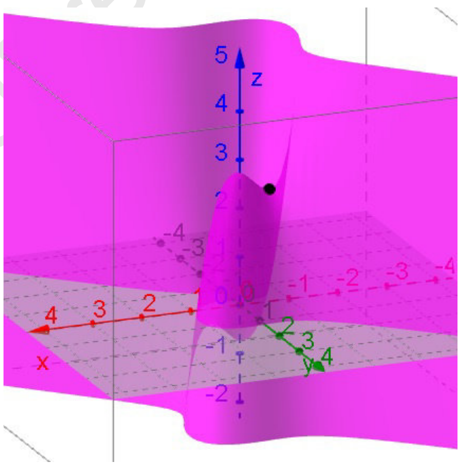

z
x
y

$$III. 2z + 2 = 0.$$

From III. we have that $z = -1$. From II. we get that $y = -6x$, thus I. can be written as $3x^2 - 72x = 0$. Thus $x_1 = 0$ or $x_2 = 24$. Because of $y = -6x$ the stationary points are $\underline{r}_1 = (0, 0, -1)$ and $\underline{r}_2 = (24, -144, -1)$.
step 3:

$$f''(x, y, z) = \begin{pmatrix} 6x & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

step 4:

$$\Delta_3(\underline{r}_1) = \begin{vmatrix} 0 & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} \qquad \Delta_3(\underline{r}_2) = \begin{vmatrix} 144 & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

step 5: We have no theorem to decide the case of $\underline{r}_1 = (0, 0, -1)$, because $\Delta_1(\underline{r}_1) = 0$.

The case of $\underline{r}_2 = (24, -144, -1)$:

$$\Delta_1(\underline{r}_2) = \partial_1\partial_1 f(\underline{r}_2) = 144 > 0$$

$$\Delta_2(\underline{r}_2) = det \begin{pmatrix} 144 & 12 \\ 12 & 2 \end{pmatrix} = 2 \cdot 144 - 12 \cdot 12 = 144 > 0$$

$$\Delta_3(\underline{r}_2) = det \begin{pmatrix} 144 & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \cdot det \begin{pmatrix} 144 & 12 \\ 12 & 2 \end{pmatrix} = 288 > 0.$$

$\Rightarrow$ A)
step 6: f has local minimum at $\underline{r}_2 = (24, -144, -1)$.

The investigation of $\underline{r}_1 = (0, 0, -1)$

$$f(x,y,z) = x^3 + y^2 + z^2 + 12xy + 2z$$

$$f(\underline{r}_1) = f(0,0,-1) = -1, \quad f(x,0,-1) = x^3 - 1$$

$$x < 0 \;\Rightarrow x^3 < 0 \;\Rightarrow\; x^3 - 1 < -1$$

$$x > 0 \;\Rightarrow x^3 > 0 \;\Rightarrow\; x^3 - 1 > -1$$

$$\Rightarrow f \text{ has no local extrema at } \underline{r}_1.$$

Exercise 4.1.16 *A flat disc is presented by the inequality* $x^2 + y^2 \leq 1$*. The disc is heated such way, that the temperature in the point* (x,y) *is* $T(x,y) = x^2 + 2y^2 - x$*. Find the hottest and coldest points of the disc.*

Solution. We are looking for the global extrema of the function

$$T(x,y) = x^2 + 2y^2 - x$$

subject to the constraint $x^2 + y^2 \leq 1$, i.e. over the unit disc centered at the origin.

$$(D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\})$$

D is a bounded and closed set, so the continuous function T attains its minimum and maximum over D; the extreme values are attained in the interior of D (D^o) or on the boundary of D $x^2 + y^2 = 1$.
<u>I. Extreme values over D^o:</u>

$$\partial_1 T(x,y) = 2x - 1 \quad \partial_2 T(x,y) = 4y$$

$$I. \;\; 2x - 1 = 0$$

$$II. \;\; 4y = 0$$

The only solution of this system is $x = \frac{1}{2},\ y = 0$, thus the only stationary point is $\underline{r}_0 = \left(\frac{1}{2}, 0\right) \in D^o$. The second derivative:

$$\partial_1\partial_1 T(x,y) = 2 \quad \partial_1\partial_2 T(x,y) = 0$$

$$\partial_2\partial_1 T(x,y) = 0 \qquad \partial_2\partial_2 T(x,y) = 4$$

Since

$$\partial_1\partial_1 T(\underline{r}_0) = 2 > 0 \quad \text{and} \quad det\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = 8 - 0 > 0,$$

T has a local minimum at the point $\underline{r}_0$ and $T(\underline{r}_0) = T\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$.

II. Extreme values on the boundary of D:

We are looking for the extreme values of $T(x,y) = x^2 + 2y^2 - x$ subject to the constraint $x^2 + y^2 = 1$, i.e. on the unit circle K centered at the origin.

$$(K = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\})$$

K is a bounded and closed set, so the continuous function T attains its minimum and maximum over K. Since

$$x^2 + y^2 = 1 \;\Rightarrow\; y^2 = 1 - x^2,$$

thus we need to find the extreme values of $t(x) := x^2 + 2(1 - x^2) - x = -x^2 - x + 2, \;\; x \in [-1, 1]$ over the closed interval $[-1, 1]$.

$\star$ extreme values over $]-1, 1[$:

$t'(x) = -2x - 1$, $t'(x) = 0 \Leftrightarrow \; x = -\frac{1}{2}$

$t''\left(-\frac{1}{2}\right) = -2 < 0 \;\Rightarrow\; x = -\frac{1}{2}$ is a local maximizer for$t$

Since $y = \pm\sqrt{1 - x^2}$, thus with $x = -\frac{1}{2}$ we have $y = \pm\sqrt{1 - \frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$.

$$\text{Then } \; T\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{9}{4}.$$

$\star$ values of the function at the endpoints of $[-1, 1]$:

$$x = -1 \;\Rightarrow y = 0 \text{ and } \; T(-1, 0) = 2$$

$$x = 1 \;\Rightarrow y = 0 \text{ and } \; T(1, 0) = 0$$

All together among the values

$$T(-1,0)=2, \quad T(1,0)=0, \quad T\left(\frac{1}{2},0\right)=-\frac{1}{4}$$

$$T\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)=T\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)=\frac{9}{4}$$

we can find the maximum and minimum of the temperature T. The coldest point of the disc: $\left(\frac{1}{2},0\right)$, the hottest points of the disc: $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.

4.2 Curves

Exercise 4.2.1 *Consider the following helix:*

$$\underline{r}(t)=R\cdot\cos t\cdot\underline{i}+R\cdot\sin t\cdot\underline{j}+\lambda\cdot t\cdot\underline{k} \quad (t\in\mathbb{R})$$

Calculate the angle of the axis of the cylinder containing the helix and the tangent line of the helix at t.

Solution: The direction of the axis of the cylinder and the vector $\underline{k}=(0,0,1)$ are the same. The direction of the tangent line at t:

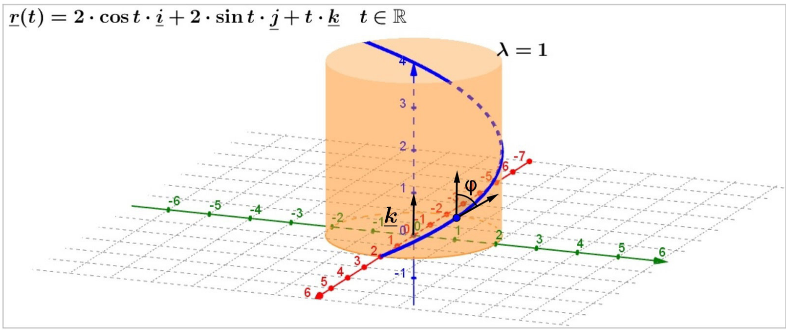
$$\underline{r}'(t)=(-R\cdot\sin(t),R\cdot\cos(t),\lambda)$$

If φ denotes the angle of the vectors $\underline{a}$ and $\underline{b}$, then φ can be calculated by the scalar product of $\underline{a}$ and $\underline{b}$: $\cos\varphi=\frac{\underline{a}\bullet\underline{b}}{|\underline{a}|\cdot|\underline{b}|}$. Thus the angle of $\underline{k}$ and $\underline{r}'(t)$: $\varphi=\arccos\left(\frac{\underline{k}\bullet\underline{r}'(t)}{|\underline{k}|\cdot|\underline{r}'(t)|}\right)$.

So $\varphi=\frac{0\cdot(-R)\cdot\sin(t)+0\cdot R\cdot\cos(t)+1\cdot\lambda}{\sqrt{(-R\cdot\sin(t))^2+(R\cdot\cos(t))^2+\lambda^2}}=\underline{\underline{\arccos\left(\frac{\lambda}{\sqrt{R^2+\lambda^2}}\right)}}=\text{constant}.$

Exercise 4.2.2 *What is the type of the following curve:*

$$\underline{r}(t)=3\cos t\cdot\underline{i}+3\sin t\cdot\underline{j}+(2-3\cos t-3\sin t)\cdot\underline{k}$$



Solution: The image of the curve is on a cylinder where the radius 3. If we denote the points of the curve with (x, y, z), then we have

$$(\star) \qquad z = 2 - x - y, \quad \text{i.e.} \quad x + y + z - 2 = 0.$$

Remember, that the equation of the plane which passes through the point $P_0 = (x_0, y_0, z_0)$ and perpendicular with the vector $\underline{n} = (n_1, n_2, n_3)$:

$$n_1 \cdot (x - x_0) + n_2 \cdot (y - y_0) + n_3 \cdot (z - z_0) = 0$$

So we get that $(\star)$ is the equation of a plane which perpendicular with $(1, 1, 1)$. Thus the plene of the investigated curve is not perpendicular with the axis of the cylinder. It means that this curve is an ellipse.

Exercise 4.2.3 *Give the parabola $y = x^2$ by a vector-valued function. Is there extremas of the curvature function? If there is, minimum or maximum?*

Solution: $\underline{r}(t) = t \cdot \underline{i} + t^2 \cdot \underline{j} \quad (t \in \mathbb{R}) \;\Rightarrow\; \underline{r}'(t) = 1 \cdot \underline{i} + 2t \cdot \underline{j} \;\Rightarrow\; \underline{r}''(t) = 0 \cdot \underline{i} + 2 \cdot \underline{j}$
The curvature function:

$$\kappa(t) = \frac{|\underline{r}'(t) \times \underline{r}''(t)|}{|\underline{r}'(t)|^3} \qquad (t \in \mathbb{R})$$

$$\underline{r}'(t) \times \underline{r}''(t) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} =$$

$$= +\underline{i} \cdot \underbrace{(2t \cdot 0 - 0 \cdot 2)}_{0} - \underline{j} \cdot \underbrace{(1 \cdot 0 - 0 \cdot 0)}_{0} + \underline{k} \cdot \underbrace{(1 \cdot 2 - 2t \cdot 0)}_{2}$$

$$\kappa(t) = \frac{\sqrt{0^2 + 0^2 + 2^2}}{\left(\sqrt{1^2 + (2t)^2 + 0^2}\right)^3} = \frac{2}{\left(\sqrt{1 + 4t^2}\right)^3}$$

If the value is $1 + 4t^2$ the least, then the curvature is the greatest. This is the case $t = 0$. So the curvature function has maximum at $t = 0$ (at the vertex).

The curvature function has no minimum, because the function $t \to 1 + 4t^2$ $(t \in \mathbb{R})$ has no maximum.

Exercise 4.2.4 *The position-time function of a particle can be given by* $\underline{r}(t) = (\cos t, \sin t, t), \quad t \in [0, 4\pi]$.

- *Give the velocity-time function of the moving particle.*
- *Give the acceleration-time function of the moving particle.*
- *Give the scalar velocity of the moving particle at* $t_0 = \pi$. *(the magnitude of velocity vector* $\underline{v}(t_0)$*)*
- *Give the moments when the velocity vector and the acceleration vector are perpendicular.*
- *Calculate the covered distance on the* $[0, \pi]$ *time interval.*
- *Calculate the curvature at* $t = \pi$.
- *Give the tangent line of curve* $\underline{r}(t)$ *at* $t_0 = \pi$. *By the help of the tangent line give an approximation for the position of the particle when* $t = 3$.
- *Give the torsion of curve* $\underline{r}(t)$ *at* $t = \pi$.
- *Determine the equation of osculating plane of curve* $\underline{r}(t)$ *at* $t = \pi$.

Solution:

- velocity-time function:

$$\underline{r}(t) = (\cos t, \sin t, t) \quad \Rightarrow \quad \underline{v}(t) = \underline{r}'(t) = (-\sin t, \cos t, 1)$$

- acceleration-time function

$$\underline{a}(t) = \underline{v}'(t) = \underline{r}''(t) = (-\cos t, -\sin t, 0)$$

- $\underline{v}(\pi) = \underline{r}'(\pi) = (-\sin \pi, \cos \pi, 1) = (0, -1, 1)$

$$v(\pi) = |\underline{r}'(\pi)| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$$

- the time instants when the velocity vector and the acceleration vector are perpendicular:

$$\underline{v}(t) \bullet \underline{a}(t) = 0$$

$(-\sin t, \cos t, 1) \bullet (-\cos t, -\sin t, 0) = 0$, i.e.

$$(-\sin t) \cdot (-\cos t) + (\cos t) \cdot (-\sin t) + 1 \cdot 0 = 0 \;\Leftrightarrow\; 0 = 0.$$

It means that each time instants satisfies the requirement property.

- the covered distance between the time instants $t_1 = 0$ and $t_2 = \pi$ is the same as the arc length of the curve $\underline{r}(t)$:

$$L = \int_{t_1}^{t_2} |r'(t)| \,\mathrm{d}t \quad \left(= \int_{t_1}^{t_2} v(t) \,\mathrm{d}t \right)$$

(Note, that $v(t) = \sqrt{2}$)

$$L = \int_0^\pi |(-\sin t, \cos t, 1)| \,\mathrm{d}t = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} \,\mathrm{d}t =$$

$$= \int_0^\pi \sqrt{2} \,\mathrm{d}t = \left[\sqrt{2} \cdot t \right]_0^\pi = \sqrt{2} \cdot \pi - \sqrt{2} \cdot 0 = \sqrt{2} \cdot \pi.$$

- The curvature at $t = \pi$:

$$\kappa(t) = \frac{|\underline{r}'(t) \times \underline{r}''(t)|}{|\underline{r}'(t)|^3} \quad \kappa(\pi) = \frac{|\underline{r}'(\pi) \times \underline{r}''(\pi)|}{|\underline{r}'(\pi)|^3}$$

$\underline{r}(t) = (\cos t, \sin t, t)$, $\underline{r}'(t) = (-\sin t, \cos t, 1)$
$\underline{r}''(t) = (-\cos t, -\sin t, 0)$, thus

$$\underline{r}'(\pi) = (0, -1, 1), \quad \underline{r}''(\pi) = (1, 0, 0) \;\Rightarrow$$

$$\underline{r}'(\pi) \times \underline{r}''(\pi) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0 \cdot \underline{i} + 1 \cdot \underline{j} + 1 \cdot \underline{k}$$

$$\underline{\underline{\kappa(\pi)}} = \frac{\sqrt{0^2 + 1^2 + 1^2}}{(\sqrt{0^2 + (-1)^2 + 1^2})^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \underline{\underline{\frac{1}{2}}}$$

- the tangent line of the curve $\underline{r}(t)$ at $t_0 = \pi$:

$$\underline{e}(t) = \underline{r}(t_0) + \underline{r}'(t_0) \cdot (t - t_0);$$

$\underline{r}(t) \approx \underline{e}(t)$. As $\underline{r}(t) = (\cos t, \sin t, t)$, thus $\underline{r}(\pi) = (-1, 0, \pi)$. We already know, that $\underline{r}'(\pi) = (0, -1, 1)$. So we have

$$\underline{e}(t) = \begin{pmatrix} -1 \\ 0 \\ \pi \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot (t - \pi) = \begin{pmatrix} -1 + 0 \cdot (t - \pi) \\ 0 + (-1) \cdot (t - \pi) \\ \pi + 1 \cdot (t - \pi) \end{pmatrix}, \text{ azaz}$$

$$\underline{e}(t) = \begin{pmatrix} -1 \\ \pi - t \\ t \end{pmatrix} \qquad \underline{r}(3) \approx \underline{e}(3) = \begin{pmatrix} -1 \\ 0.14 \\ 3 \end{pmatrix}$$

- The torsion at $t = \pi$:

$$\tau(t) = \frac{\underline{r}'(t)\underline{r}''(t)\underline{r}'''(t)}{|\underline{r}'(t) \times \underline{r}''(t)|^2},$$

so

$$\tau(\pi) = \frac{\underline{r}'(\pi)\underline{r}''(\pi)\underline{r}'''(\pi)}{|\underline{r}'(\pi) \times \underline{r}''(\pi)|^2}$$

$$\underline{r}'''(t) = (\sin t, -\cos t, 0) \text{ and } \underline{r}'''(\pi) = (0,1,0).$$

By the definition of the mixed product

$$\underline{r}'(\pi)\underline{r}''(\pi)\underline{r}'''(\pi) = \left(\underline{r}'(\pi) \times \underline{r}''(\pi)\right) \bullet \underline{r}'''(\pi) = (0,1,1) \bullet (0,1,0) = 1,$$

thus $\tau(\pi) = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$.

- Osculating plane at $t = \pi$:

  A point of the plane:

  $$P_0 = \underline{r}(\pi) = (\cos\pi, \sin\pi, \pi) = (-1, 0, \pi)$$

  A normal vector of the plane: $\underline{B}(\pi)$. As $\underline{B}(\pi) \parallel \underline{r}'(\pi) \times \underline{r}''(\pi)$ thus $\underline{n} = \underline{r}'(\pi) \times \underline{r}''(\pi)$ $(= (0,1,1).)$ The equation of the plane:

  $$0 \cdot (x+1) + 1 \cdot (y-0) + 1 \cdot (z-\pi) = 0, \quad \text{i.e. } \underline{\underline{y + z - \pi = 0}}.$$

4.3 Surfaces and vector fields

Exercise 4.3.1 *Consider the surface determined by*

$$(u,v) \mapsto \underline{r}(u,v) = \left(u^2 - 2v^2, uv^2, u^2v - u\right) \qquad (u,v) \in \mathbb{R}^2.$$

Give the equation of the tangent plane which passes through on the point $\underline{r}(u_0,v_0)$, if $(u_0,v_0) = (2,-1)$.

Solution: The investigated plane passes through on the point

$$\underline{r}(u_0,v_0) = \underline{r}(2,-1) = \left(4 - 2 \cdot 1, 2 \cdot (-1)^2, 4 \cdot (-1) - 2\right) = (2,2,-6).$$

A normal vector of the tangent plane when $(u_0, v_0) = (2, -1)$ is

$$\underline{n}_{(2,-1)} = \frac{\partial \underline{r}}{\partial u}(2,-1) \times \frac{\partial \underline{r}}{\partial v}(2,-1).$$

Here

$$\frac{\partial \underline{r}}{\partial u}(u,v) = (2u, v^2, 2uv-1) \text{ and } \frac{\partial \underline{r}}{\partial v}(u,v) = (-4v, 2uv, u^2),$$

thus $\underline{n}_{(2,-1)} = (4,1,-5) \times (4,-4,4)$, i.e.

$$\underline{n}_{(2,-1)} = \left| \begin{array}{ccc|cc} \underline{i} & \underline{j} & \underline{k} & \underline{i} & \underline{j} \\ 4 & 1 & -5 & 4 & 1 \\ 4 & -4 & 4 & 4 & -4 \end{array} \right. = (-16, -36, -20).$$

The equation of the investigated plane is $-16(x-2)-36(y-2)-20(z+6) = 0$, i.e.

$$4(x-2) + 9(y-2) + 5(z+6) = 0.$$

Exercise 4.3.2 *Calculate the derivative, divergence and curl of the vector field*

$$\underline{v}(x,y,z) = \left(x^2 - y^2, y^2 - z^2, z^2 - x^2\right)$$

at the point $\underline{r}_0 = (3, 2, -1)$.

Solution. $\underline{v}'(\underline{r}) = \underline{v}'(x,y,z) = \begin{pmatrix} 2x & -2y & 0 \\ 0 & 2y & -2z \\ -2x & 0 & 2z \end{pmatrix} \Rightarrow$

$\underline{v}'(\underline{r}_0) = \underline{v}'(3,2,-1) = \begin{pmatrix} 6 & -4 & 0 \\ 0 & 4 & 2 \\ -6 & 0 & -2 \end{pmatrix}.$

$$\text{div } \underline{v} = (\partial_1, \partial_2, \partial_3) \bullet (v_1, v_2, v_3) = \nabla \bullet \underline{v}$$

Here

$$\text{div } \underline{v}(\underline{r}_0) = \partial_1 v_1(\underline{r}_0) + \partial_2 v_2(\underline{r}_0) + \partial_3 v_3(\underline{r}_0) = 6 + 4 - 2 = 8.$$

On the other hand

$$\operatorname{rot} \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \nabla \times \underline{v}$$

that is

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{matrix} \underline{i} & \underline{j} \\ \partial_1 & \partial_2 \\ v_1 & v_2 \end{matrix} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}.$$

Thus

$$\operatorname{rot} \underline{v}(\underline{r}_0) = \begin{pmatrix} \partial_2 v_3(\underline{r}_0) - \partial_3 v_2(\underline{r}_0) \\ \partial_3 v_1(\underline{r}_0) - \partial_1 v_3(\underline{r}_0) \\ \partial_1 v_2(\underline{r}_0) - \partial_2 v_1(\underline{r}_0) \end{pmatrix} = \begin{pmatrix} 0-2 \\ 0-(-6) \\ 0-(-4) \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix}.$$

5

Integration for Multivariable functions

5.1 Double and triple integrals on intervals

Let $I = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ be a closed two-dimensional interval. If $f\colon I \to \mathbb{R}$ is continuous, then the double integral of f over I can be calculated as two successive integration of functions with one variable as follows:

$$\iint_I f(x,y)\,dxdy = \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x,y)\,dx \right)\, dy$$

or

$$\iint_I f(x,y)\,dxdy = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x,y)\,dy \right)\, dx$$

.

Exercise 5.1.1 *Calculate the integral of the function* $f(x,y) = \frac{x}{y}$ *on the two dimensional interval*

$$D = \{(x,y)|1 \le x \le 2; 3 \le y \le 4\}$$

(i.e. $D = [1,2] \times [3,4]$*).*

Solution.

$$\int_{x=1}^{2}\left(\int_{y=3}^{4} f(x,y)\,dy\right)dx = \int_{x=1}^{2}\left(\int_{y=3}^{4} \frac{x}{y}\,dy\right)dx =$$

$$= \int_{x=1}^{2}\left(\int_{y=3}^{4} x\frac{1}{y}\,dy\right)dx = \int_{x=1}^{2}\left(x\int_{y=3}^{4} \frac{1}{y}\,dy\right)dx =$$

$$= \int_{x=1}^{2}\left(x\Big[ln|y|\Big]_{y=3}^{4}\right)dx = \int_{x=1}^{2}\left(x\Big(ln4 - ln3\Big)\right)dx = ln\frac{4}{3}\int_{x=1}^{2} x\,dx =$$

$$= ln\frac{4}{3}\left[\frac{x^2}{2}\right]_{x=1}^{2} = \left(ln\frac{4}{3}\right)\left(\frac{2^2}{2} - \frac{1^2}{2}\right) = \frac{3}{2}ln\frac{4}{3}$$

Exercise 5.1.2 *Calculate the integral of the function*

$$f(x,y) = \sin x + 2\sin y$$

on the two dimensional interval

$$D = \{(x,y)| -\frac{\pi}{2} \le x \le \frac{\pi}{2}; 0 \le y \le \frac{\pi}{2}\}.$$

Solution.

$$\int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_{y=0}^{\frac{\pi}{2}} (\sin x + 2\sin y)\,dy\right)dx =$$

$$= \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_{y=0}^{\frac{\pi}{2}} \sin x\,dy + \int_{y=0}^{\frac{\pi}{2}} 2\sin y\,dy\right)dx =$$

$$= \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}}\left([y\cdot\sin x]_{y=0}^{\frac{\pi}{2}} + 2\cdot[-\cos y]_{y=0}^{\frac{\pi}{2}}\right)dx =$$

$$= \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi}{2}\sin x - 0\cdot\sin x + 2(-\cos\frac{\pi}{2} - (-\cos 0))\right)dx =$$

$$= \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi}{2}\sin x + 2\right)dx = [-\frac{\pi}{2}\cos x + 2x]_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} = (0+\pi) - (0-\pi) = 2\pi.$$

Remark 5.1.1 *Let $I = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$ be a closed 3-dimensional interval. If $f\colon I \to \mathbb{R}$ is continuous, then its triple integral over I can be calculated as three successive integration of functions with one variable:*

$$\iiint\limits_I f(x,y,z)\,dx\,dy\,dz = \int\limits_{a_3}^{b_3} \left(\int\limits_{a_2}^{b_2} \left(\int\limits_{a_1}^{b_1} f(x,y,z)\,dx \right) dy \right) dz.$$

Exercise 5.1.3 *Calculate the integral of the function*

$$f(x,y,z) = x^3 z - 6yz^2 + 4z - 2$$

on the three dimensional interval $I = [1,2] \times [0,3] \times [2,6]$.

Solution. If the function f is continuous on the three dimensional interval I then we can calculate in the following way:

$$\int_{x=1}^{2} \left(\int_{y=0}^{3} \left(\int_{z=2}^{6} f(x,y,z)\,dz \right) dy \right) dx$$

Thus

$$\int_{x=1}^{2} \left(\int_{y=0}^{3} \left(\int_{z=2}^{6} x^3 z - 6yz^2 + 4z - 2\,dz \right) dy \right) dx =$$

$$= \int_{x=1}^{2} \left(\int_{y=0}^{3} \left[x^3 \frac{z^2}{2} - 6y \frac{z^3}{3} + 4 \frac{z^2}{2} - 2z \right]_{z=2}^{6} dy \right) dx =$$

$$= \int_{x=1}^{2} \left(\int_{y=0}^{3} (18x^3 - 432y + 72 - 12) - (2x^3 - 16y + 8 - 4)\,dy \right) dx =$$

$$\int_{x=1}^{2} \left(\int_{y=0}^{3} 16x^3 - 416y + 56\,dy \right) dx =$$

$$= \int_{x=1}^{2} \left[16x^3 y - 416 \frac{y^2}{2} + 56y \right]_{y=0}^{3} dx =$$

$$\int_{x=1}^{2} 48x^3 - 208 \cdot 9 + 168\,dx = \int_{x=1}^{2} 48x^3 - 1704\,dx =$$

$$= \left[48\frac{x^4}{4} - 1704x\right]_{x=1}^{2} =$$

$$= 192 - 3408 - (12 - 1704) = -1524.$$

5.2 Integrals over general regions

If $f_1, f_2\colon [a,b] \to \mathbb{R}$ are continuous functions with $f_1(x) \le f_2(x),\ x \in [a,b]$, then

$$H = \{(x,y) \in \mathbb{R}^2 | a \le x \le b,\ f_1(x) \le y \le f_2(x)\}$$

is called a normal domain with respect to the x-axis (or normal domain of type 1). In this case we have

$$\int\int_H f(x,y)\,dxdy = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right)\,dx$$

If $g_1, g_2\colon [a,b] \to \mathbb{R}$ are continuous function with $g_1(y) \le g_2(y),\ y \in [a,b]$, then

$$H = \{(x,y) \in \mathbb{R}^2 | a \le y \le b,\ g_1(y) \le x \le g_2(y)\}$$

is called a normal domain with respect to the y-axis (or normal domain of type 2). In this case we have

$$\int\int_H f(x,y)\,dxdy = \int_a^b \left(\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\right)\,dy$$

Exercise 5.2.1 *(Integral over normal). Calculate the integral of*

$$f(x,y) = 2xy$$

on the set $D = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le 1\}$.

Solution. Because D is a normal domain of type 1, thus

$$\int_{x=0}^{1}\left(\int_{y=x^2}^{1} 2xy\,dy\right)dx=\int_{x=0}^{1}\left[2x\frac{y^2}{2}\right]_{y=x^2}^{1}dx=\int_{x=0}^{1}\left[xy^2\right]_{y=x^2}^{1}dx=$$

$$=\int_{x=0}^{1}\left(x\cdot 1^2-x\cdot(x^2)^2\right)dx=\int_{x=0}^{1}\left(x-x^5\right)dx=\left[\frac{x^2}{2}-\frac{x^6}{6}\right]_{x=0}^{1}=\frac{1}{3}.$$

D can be written as a normal domain of type 2:

$$D=\{(x,y)\mid 0\le y\le 1,\ 0\le x\le\sqrt{y}\},$$

thus

$$\int_{y=0}^{1}\left(\int_{x=0}^{\sqrt{y}} 2xy\,dx\right)dy=\int_{y=0}^{1}\left[2\frac{x^2}{2}y\right]_{x=0}^{\sqrt{y}}dy=$$

$$=\int_{y=0}^{1}\left(2\frac{\sqrt{y}^2}{2}y-2\frac{0^2}{2}y\right)dy=$$

$$=\int_{y=0} y^2\,dy=\left[\frac{y^3}{3}\right]_{y=0}^{1}=\frac{1}{3}.$$

Remark 5.2.1 *(Integrals in polar coordiantes) Let*

$$\underline{h}(r,\alpha)=(r\cdot\cos\alpha,r\cdot\sin\alpha)$$

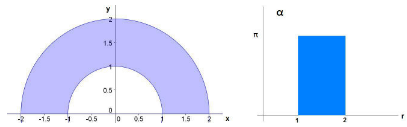
for all $r,\alpha\in\mathbb{R}$. *If* $H\subset[0,\infty)\times[0,2\pi]$ *is a two-dimensional closed interval and* $f\colon\underline{h}(H)\to\mathbb{R}$ *is integrable over* $h(H)$, *then*

$$\int_{\underline{h}(H)} f(x,y)\,dx\,dy=\int_{H} f(r\cdot\cos\alpha,r\cdot\sin\alpha)\cdot r\,dr\,d\alpha.$$

Exercise 5.2.2 *Calculate the integral of*

$$f(x,y)=ln\ (x^2+y^2)$$

over the set $T=\{(x,y)\mid 1\le x^2+y^2\le 4,\ 0\le y\}$.



Solution. Here $T = \underline{h}(H)$ is a circular domain.

$$H = \{(r, \alpha) \mid 1 \le r \le 2,\ 0 \le \alpha \le \pi\}.$$

Because of

$$x^2 + y^2 = r^2 \cos^2 \alpha + r^2 \sin^2 \alpha = r^2(\cos^2 \alpha + \sin^2 \alpha) = r^2$$

we get that

$$\int\int_T f(x, y)\,dx\,dy = \int_{\alpha=0}^{\pi} \left(\int_{r=1}^{2} r\ln\, r^2\,dr \right) d\alpha =$$

$$= \int_{\alpha=0}^{\pi} \left(\int_{r=1}^{2} 2r\ln\, r\,dr \right) d\alpha = 2 \int_{\alpha=0}^{\pi} \left(\int_{r=1}^{2} r\ln\, r\,dr \right) d\alpha.$$

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'$$

Let $g(r) = \ln\, r$ and $f'(r) = r$. Then $g'(r) = \frac{1}{r}$ and $f(r) = \frac{r^2}{2}$. Thus

$$\int_{r=1}^{2} r\ln\, r\,dr = \left[\frac{r^2}{2} \ln\, r \right]_1^2 - \int_{r=1}^{2} \frac{r^2}{2} \frac{1}{r}\,dr = 2\ln\, 2 - \frac{1}{2}\ln\, 1 - \int_{r=1}^{2} \frac{r}{2}\,dr =$$

$$= 2\ln\, 2 - \frac{1}{2} \left[\frac{r^2}{2} \right]_1^2 = 2\ln\, 2 - \frac{1}{2}\left(2 - \frac{1}{2}\right) = 2\ln\, 2 - \frac{3}{4}.$$

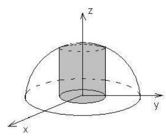
$$2 \int_{\alpha=0}^{\pi} \left(\int_{r=1}^{2} r\ln\, r\,dr \right) d\alpha = 2 \int_{\alpha=0}^{\pi} 2\ln\, 2 - \frac{3}{4}\,d\alpha = 2\left[\alpha\left(2\ln\, 2 - \frac{3}{4} \right) \right]_{\alpha=0}^{\pi} =$$

$$= 2\left(2\ln\, 2 - \frac{3}{4} \right)\pi.$$

Exercise 5.2.3 *Calculate the integral of*

$$f(x,y)=\sqrt{4-x^2-y^2}$$

over the set $T=\{(x,y)|\ x^2+y^2\le 1\}$. *(i.e. calculate the volume of the body enclosed by a cylinder and a hemisphere.)*



Solution. With substitutions $x=r\cos\alpha$, $y=r\sin\alpha$ we have that

$$H=\{(r,\alpha)\mid 0\le r\le 1,\ 0\le\alpha\le\ 2\pi\}$$

therefore

$$\int\int_T f(x,y)\,dx\,dy=\int_{\alpha=0}^{2\pi}\left(\int_{r=0}^{1} r\sqrt{4-r^2}\,dr\right)d\alpha=$$

$$=-\frac{1}{2}\int_{\alpha=0}^{2\pi}\left(\int_{r=0}^{1}(-2)r\sqrt{4-r^2}\,dr\right)d\alpha=-\frac{1}{2}\int_{\alpha=0}^{2\pi}\left[\frac{(4-r^2)^{3/2}}{(3/2)}\right]_{r=0}^{1}d\alpha=$$

$$=-\frac{1}{2}\int_{\alpha=0}^{2\pi}\left(3^{3/2}-4^{3/2}\right)\frac{2}{3}\,d\alpha=-\frac{1}{2}\left(3^{3/2}-4^{3/2}\right)\frac{2}{3}\int_{\alpha=0}^{2\pi}1\,d\alpha=$$

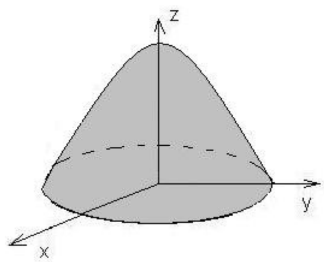
$$-\frac{1}{2}\left(3^{3/2}-4^{3/2}\right)\frac{2}{3}[\alpha]_0^{2\pi}=-\frac{2\pi}{3}\left(3^{3/2}-4^{3/2}\right)=$$

$$=-\frac{2\pi}{3}\left(\sqrt{27}-8\right)=\frac{2\pi}{3}\left(8-\sqrt{27}\right).$$

Exercise 5.2.4 *Calculate the volume of the body enclosed by the paraboloid*

$$z = 8 - 2x^2 - 2y^2$$

and the plane $z = 0$.



Solution. Let $f(x, y) := 8 - 2x^2 - 2y^2$. The equation $0 = 8 - 2x^2 - 2y^2$ implies that $x^2 + y^2 = 4$. So we have to integrate over the, where the center of the disk is the origin and the radius is 2: $T = \{(x, y)|\ x^2 + y^2 \le 4\}$ As

$$H = \{(r, \alpha) \mid 0 \le r \le 2,\ 0 \le \alpha \le\ 2\pi\}$$

therefore with substitution $x = r\cos\alpha$, $y = r\sin\alpha$ we get that

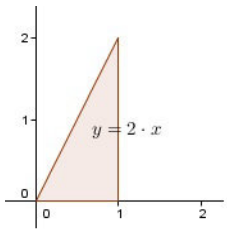
$$\int\int_T f(x, y)\, dx\, dy = \int_{\alpha=0}^{2\pi} \left(\int_{r=0}^{2} (8 - 2r^2) r\, dr \right) d\alpha =$$

$$= \int_{\alpha=0}^{2\pi} \left(\int_{r=0}^{2} 8r - 2r^3\, dr \right) d\alpha = \int_{\alpha=0}^{2\pi} \left[8\frac{r^2}{2} - 2\frac{r^4}{4} \right]_{r=0}^{2} d\alpha =$$

$$\int_{\alpha=0}^{2\pi} 16 - 8\, d\alpha = [8\alpha]_{\alpha=0}^{2\pi} = 16\pi.$$

5.3 Applications

Exercise 5.3.1 *Calculate the mass of the inhomogeneous thin plate if density of the plate: $\rho(x,y) = 6x + 6y + 6$ $[g/cm^2]$. Give the coordinates of the center of the mass with the help of density.*



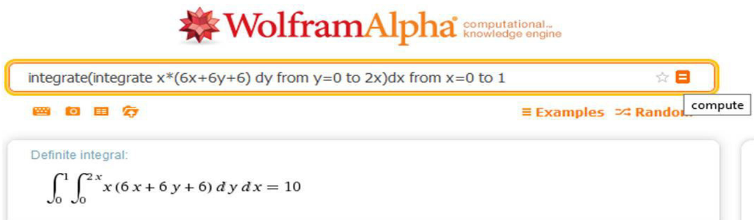
Solution. $m = \int\int_H \rho(x,y)\,dxdy = \int_0^1 \left(\int_0^{2x} 6x + 6y + 6\,dy\right)\,dx =$

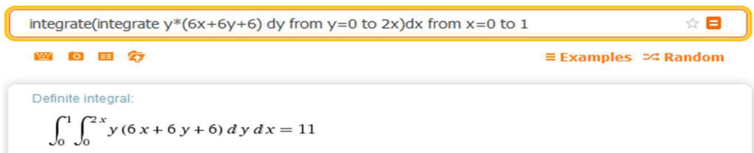
$= \int_0^1 \left[6xy + 6\frac{y^2}{2} + 6y\right]_0^{2x} dx = \int_0^1 6x \cdot 2x + 6 \cdot \frac{(2x)^2}{2} + 6 \cdot 2x\,dx =$

$= \int_0^1 12x^2 + 12x^2 + 12x\,dx = \left[24 \cdot \frac{x^3}{3} + 12 \cdot \frac{x^2}{2}\right]_0^1 = 8 + 6 - 0 = 14$ $[g]$.

$x_S = \frac{\int\int_H x\cdot\rho(x,y)\,dxdy}{\int\int_H \rho(x,y)\,dxdy} \quad y_S = \frac{\int\int_H y\cdot\rho(x,y)\,dxdy}{\int\int_H \rho(x,y)\,dxdy}$;

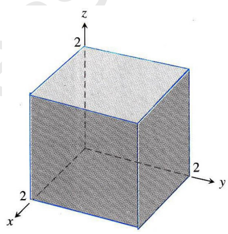
i.e here: $x_S = \frac{\int\int_H x\cdot(6x+6y+6)\,dxdy}{14} \quad y_S = \frac{\int\int_H y\cdot(6x+6y+6)\,dxdy}{14}$.





Thus $x_S = \frac{10}{14}$ and $y_S = \frac{11}{14}$.

Exercise 5.3.2 *Calculate the mass of the inhomogeneous cube if density of the cube:* $\rho(x,y,z) = x^2 + y$ $[g/cm^3]$. *Give the coordinates of the centroid with the help of density.*



Solution. $H = \{(x,y,z) | 0 \le x \le 2,\ 0 \le y \le 2,\ 0 \le z \le 2\}$

$$m = \int\int\int_H \rho(x,y,z)\,dxdydz = \int_0^2 \left(\int_0^2 \left(\int_0^2 x^2 + y\,dz \right) dy \right) dx =$$
$$= \int_0^2 \left(\int_0^2 \left[(x^2+y) \cdot z \right]_0^2 dy \right) dx = \int_0^2 \left(\int_0^2 (x^2+y) \cdot 2 - 0\,dy \right) dx =$$
$$= \int_0^2 \left(\int_0^2 2x^2 + 2y\,dy \right) dx = \int_0^2 \left[2x^2 y + y^2 \right]_0^2 dx = \int_0^2 4x^2 + 4\,dx =$$
$$= \left[4 \cdot \frac{x^3}{3} + 4x \right]_0^2 = 4 \cdot \frac{8}{3} + 4 \cdot 2 - 0 = \frac{32+24}{3} = \frac{56}{3}\ [g].$$

$$x_S = \frac{\int\int\int_H x \cdot \rho(x,y,z)\,dxdydz}{\int\int\int_H \rho(x,y,z)\,dxdydz},\quad y_S = \frac{\int\int\int_H y \cdot \rho(x,y,z)\,dxdydz}{\int\int\int_H \rho(x,y,z)\,dxdydz},$$

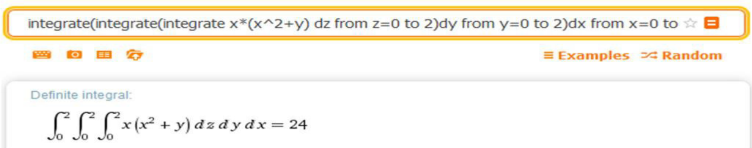
$$z_S = \frac{\int\int\int_H z \cdot \rho(x,y,z)\, dxdydz}{\int\int\int_H \rho(x,y,z)\, dxdydz}$$

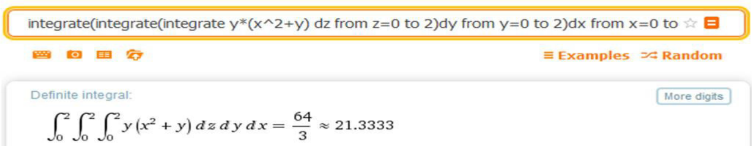
i.e. here:

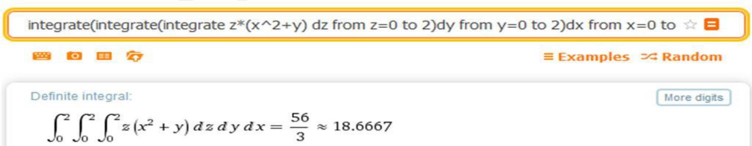
$$x_S = \frac{\int\int\int_H x \cdot (x^2+y)\, dxdydz}{\frac{56}{3}}, \quad y_S = \frac{\int\int\int_H y \cdot (x^2+y)\, dxdydz}{\frac{56}{3}},$$

$$z_S = \frac{\int\int\int_H z \cdot (x^2+y)\, dxdydz}{\frac{56}{3}}$$

Thus







$$x_S = \frac{24}{\frac{56}{3}} = \frac{72}{56} \approx 1.29, \quad y_S = \frac{\frac{64}{3}}{\frac{56}{3}} = \frac{64}{56} \approx 1.14, \quad z_S = 1$$

Remark 5.3.1 *Let $D \subset \mathbb{R}^2$ be a bounded, closed set and let $h_1, h_2\colon D \to \mathbb{R}$ be continuous functions with $h_1(x,y) \le h_2(x,y)$ for all $(x,y) \in D$. Then the set*

$$H = \{(x,y,z): \ (x,y) \in D; \ h_1(x,y) \le z \le h_2(x,y)\}$$

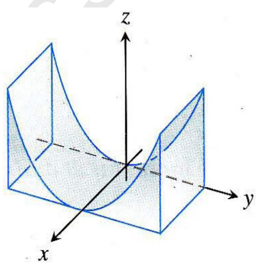
is called normal domain with respect to the xy-coordinate plane over D.

If $f\colon H \to \mathbb{R}$ is a continuous function, then

$$\int\int\int_H f = \int\int_D \left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right) dx\,dy.$$

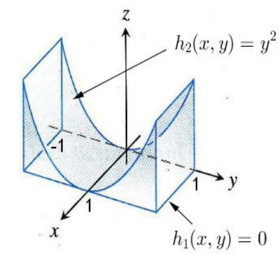
Exercise 5.3.3 *A body is enclosed by the xy coordinate plane, the cylinder $z = y^2$ and the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$ (The length is measured in [dm]). Calculate the mass, if the density function is $\rho(x,y,z) = 2x$ $[kg/dm^3]$.*

Solution.



$$m = \int\int\int_H \rho(x,y,z)\,dxdydz = \int\int_D \left(\int_{h_1(x,y)}^{h_2(x,y)} \rho(x,y,z)\,dz\right) dxdy$$

$$m = \int_0^1 \left(\int_{-1}^1 \left(\int_0^{y^2} 2x\,dz\right) dy\right) dx = \int_0^1 \left(\int_{-1}^1 [2xz]_0^{y^2}\,dy\right) dx =$$



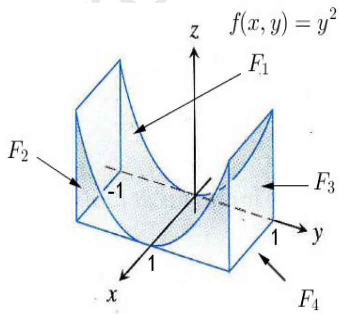
$D = [0,1] \times [-1,1]$

$$= \int_0^1 \left(\int_{-1}^1 2xy^2 dy \right) dx = \int_0^1 \left[2x\frac{y^3}{3} \right]_{-1}^1 dx = \int_0^1 2x \cdot \frac{1}{3} - 2x \cdot \left(-\frac{1}{3} \right) dx =$$

$$= \int_0^1 \frac{4}{3} x \, dx = \left[\frac{4}{3} \cdot \frac{x^2}{2} \right]_0^1 = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3} \quad [kg].$$

Exercise 5.3.4 *Calculate the surface area of the body above.*

Solution.



$(x,y) \in [0,1] \times [-1,1]$

$$F = F_1 + 2 \cdot F_2 + 2 \cdot F_3 + F_4$$

- Calculation of F_1:

$$f(x,y)=y^2, \qquad (x,y)\in[0,1]\times[-1,1]$$
$$f'_x(x,y)=0, \qquad f'_y(x,y)=2\cdot y$$
$$F_1=\int\int_{[0,1]\times[-1,1]}\sqrt{(f'_x)^2+(f'_y)^2+1}\,dx\,dy=$$
$$=\int_0^1\left(\int_{-1}^1\sqrt{(2y)^2+1}\,dy\right)dx=(\star)$$

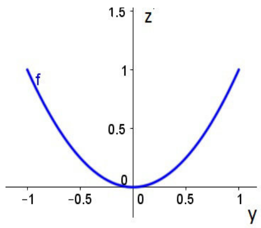
We use the rule of integral with substitution for the "inner integral".

$$\int\sqrt{(2y)^2+1}\,dy=\left[\begin{array}{c}2y=sh\,t\Rightarrow\ \ y=\frac{1}{2}sh\,t;\ \ t=arsh\,2y\\ \frac{dy}{dt}=\frac{1}{2}ch\,t\Rightarrow\ \ dy=\frac{1}{2}ch\,t\,dt\end{array}\right]=$$
$$=\int\sqrt{(sh\,t)^2+1}\cdot\frac{1}{2}ch\,t\,dt=\left[\ ch^2t-sh^2t=1\ \right]=\frac{1}{2}\cdot\int ch^2t\,dt=$$
$$=\left[\ ch^2t=\tfrac{1}{2}+\tfrac{1}{2}ch\,2t\ \right]=\frac{1}{2}\cdot\int\frac{1}{2}+\frac{1}{2}ch\,2t\,dt=$$
$$=\frac{1}{2}\cdot\left(\frac{1}{2}t+\frac{1}{2}\cdot\frac{sh\,2t}{2}\right)+c=\frac{t}{4}+\frac{sh\,2t}{8}+c=$$
$$=\left[\ sh\,2t=2\cdot sh\,t\cdot ch\,t=2\cdot sh\,t\cdot\sqrt{1+sh^2t}\ \right]=$$
$$=\frac{arsh\,2y}{4}+\frac{2\cdot 2y\cdot\sqrt{1+(2y)^2}}{8}+c$$
$$=\frac{arsh\,2y}{4}+\frac{y\cdot\sqrt{1+(2y)^2}}{2}+c$$
$$(\star)=\int_0^1\left[\frac{arsh\,2y}{4}+\frac{y\cdot\sqrt{1+(2y)^2}}{2}\right]_{-1}^1dx=$$
$$=\int_0^1\left(\frac{arsh\,2}{4}+\frac{\sqrt{5}}{2}\right)-\left(\frac{arsh\,(-2)}{4}+\frac{-\sqrt{5}}{2}\right)dx=$$
$$=\int_0^1\frac{arsh\,2}{2}+\sqrt{5}\,dx=\left[\left(\frac{arsh\,2}{2}+\sqrt{5}\right)\cdot x\right]_0^1=\frac{arsh\,2}{2}+\sqrt{5}$$

thus

$$F_1=\frac{arsh\,2}{2}+\sqrt{5}\approx 2.9579\ [dm^2].$$

- Calculation of F_2



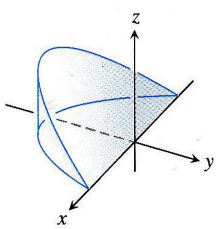
$$F_2 = \int_{-1}^{1} y^2\, dy = \left[\frac{y^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}\ [dm^2].$$

The investigated surface area:

$$F = F_1 + 2 \cdot F_2 + 2 \cdot F_3 + F_4 \approx 2.9579 + 2 \cdot \frac{2}{3} + 2 \cdot 1 + 2 \approx 8.29$$

Exercise 5.3.5 *Calculate the mass the wedge determined by the cylinder* $x^2 + y^2 = 1$ *and the planes* $z = -y$ *and* $z = 0$. *The density function is* $\rho(x, y, z) = z^2$.

Solution.



$$m = \int\int\int_H \rho(x, y, z)\, dxdydz = \int\int_D \left(\int_0^{-y} z^2\, dz\right) dxdy = (\star\star)$$

$$D = \{(r, \alpha) : \quad r \in [0, 1],\ \alpha \in [\pi, 2\pi]\}$$

$$(\star\star) = \int\int_D \left[\frac{z^3}{3}\right]_0^{-y} dxdy = \int\int_D -\frac{y^3}{3}\, dxdy = -\frac{1}{3}\int\int_D y^3\, dxdy =$$
$$= -\frac{1}{3}\int_\pi^{2\pi}\left(\int_0^1 (r\cdot\sin\alpha)^3\cdot r\, dr\right) d\alpha = -\frac{1}{3}\int_\pi^{2\pi}\left(\int_0^1 r^4\cdot\sin^3\alpha\, dr\right) d\alpha =$$
$$= -\frac{1}{3}\int_\pi^{2\pi}\left[\frac{r^5}{5}\cdot\sin^3\alpha\right]_0^1 d\alpha = -\frac{1}{3}\int_\pi^{2\pi}\frac{1}{5}\cdot\sin^3\alpha\, d\alpha =$$
$$= -\frac{1}{15}\int_\pi^{2\pi}\sin\alpha\cdot(1-\cos^2\alpha)\, d\alpha = -\frac{1}{15}\int_\pi^{2\pi}\sin\alpha - \sin\alpha\cdot\cos^2\alpha\, d\alpha =$$
$$= -\frac{1}{15}\cdot\left[-\cos\alpha + \frac{\cos^3\alpha}{3}\right]_\pi^{2\pi} =$$
$$= -\frac{1}{15}\cdot\left[\left(-1+\frac{1}{3}\right) - \left(-(-1) + \frac{-1}{3}\right)\right] = \frac{4}{15}$$

Remark 5.3.2 *We can calculate the mass in the previous example as follows:*

$$m = \int_{x=-1}^{1}\left(\int_{y=-\sqrt{1-x^2}}^{0}\left(\int_0^{-y} z^2\, dz\right) dy\right) dx$$

Exercise 5.3.6 *The position-time function of a particle can be given by* $\underline{r}(t) = (\cos t, \sin t, t), \quad t\in[0,4\pi]$. *Calculate the covered distance between the moments* $t_1 = 0$ *and* $t_2 = \pi$.

Solution. We have to calculate the arclength:

$$L = \int_{t_1}^{t_2} |r'(t)|\,\mathrm{d}t \quad \left(= \int_{t_1}^{t_2} v(t)\,\mathrm{d}t\right)$$

$$L = \int_0^\pi |(-\sin t, \cos t, 1)|\,\mathrm{d}t = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\,\mathrm{d}t =$$
$$= \int_0^\pi \sqrt{2}\,\mathrm{d}t = \left[\sqrt{2}\cdot t\right]_0^\pi = \sqrt{2}\cdot\pi - \sqrt{2}\cdot 0 = \sqrt{2}\cdot\pi.$$

<u>Remark:</u> The arclength function is $\sigma(t) = \sqrt{2}\cdot t$

5.4 Line integral, Surface integral

If G is a continuously differentiable curve parametrized by $\underline{r}(t)$; $t_1 \leq t \leq t_2$, then line integral of the continuous vector field $\underline{v}(\underline{r})$ along G is

$$\int_G \underline{v}(\underline{r})\ \mathrm{d}\underline{r} = \int_{t_1}^{t_2} \underline{v}(\underline{r}(t)) \bullet \underline{r}'(t)\ \mathrm{d}t$$

If the vector field $\underline{v}(\underline{r})$ is continuous and the surface F is parametrized by the continuously differentiable function $\underline{r} = \underline{r}(u, v)$, then the surface integral of $\underline{v}$ over F is calculated by

$$\int_F \underline{v}(\underline{r})\ \mathrm{d}f = \int\int_D \underline{v}(\underline{r}(u,v)) \bullet \left(\frac{\partial \underline{r}(u,v)}{\partial u} \times \frac{\partial \underline{r}(u,v)}{\partial v}\right) \mathrm{d}u\mathrm{d}v.$$

Divergence theorem (Gauss, Ostrogradsky): Suppose V is a compact body bounded by a closed regular surface F and the orientation of F is given by an outward-pointing normal vector field. If the vector field $\underline{v}$ is continuously differentiable in the points of an open set containing the body V, then

$$\int_F \underline{v}\ \mathrm{d}f = \int_V \operatorname{div} \underline{v}\ \mathrm{d}V.$$

The integral on the right hand side is a volume integral, where $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

Exercise 5.4.1 *Calculate the line integral of the vector field $\underline{v}(\underline{r})$ along the segment (as a curve) with endpoints A and B.*

$$\underline{v}(x, y, z) = (y + z, x + z, x + y) \qquad A(1, -2, 3),\ B(2, 1, 4)$$

Solution. The segment is the subset of the line passes through the points A and B. The directional vector of the line: $\vec{AB} = (1, 3, 1)$, thus the segment AB can be given in the following way:

$$G\colon t \to \underline{r}(t) = (1 + t, -2 + 3t, 3 + t)\,, \quad t \in [0, 1]$$

$$\int_G \underline{v}(\underline{r})\ \mathrm{d}\underline{r} = \int_0^1 \underline{v}(1 + t, -2 + 3t, 3 + t) \bullet (1, 3, 1)\,\mathrm{d}t =$$

$$= \int_0^1 \begin{pmatrix} (-2+3t)+(3+t) \\ (1+t)+(3+t) \\ (1+t)+(-2+3t) \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \, \mathrm{d}t =$$

$$= \int_0^1 \begin{pmatrix} 1+4t \\ 4+2t \\ -1+4t) \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \, \mathrm{d}t =$$

$$= \int_0^1 1 \cdot (1+4t) + 3 \cdot (4+2t) + 1 \cdot (-1+4t) \, \mathrm{d}t =$$

$$= \int_0^1 14t + 12 \, \mathrm{d}t = \left[14\frac{t^2}{2} + 12t \right]_0^1 = \left[7t^2 + 12t \right]_0^1 = 7 \cdot 1^2 + 12 \cdot 1 - 0 = 19$$

Exercise 5.4.2 *Calculate of the surface integral of* $\underline{v}$ *over* F *if* $\underline{v}(x,y,z) = (xy, z^2, y)$ *and* F *given by*

$$\underline{r}(u,v) = \left(u, uv, v^2 \right), \qquad (u,v) \in [-1,2] \times [0,1]$$

Solution. With the notation $D := [-1,2] \times [0,1]$ we have that

$$\int_F \underline{v}(\underline{r}) \, \mathrm{d}f = \iint_D \underline{v}(u, uv, v^2) \bullet \left(\frac{\partial \underline{r}(u,v)}{\partial u} \times \frac{\partial \underline{r}(u,v)}{\partial v} \right) \, \mathrm{d}u\mathrm{d}v = (\star)$$

$$\frac{\partial \underline{r}(u,v)}{\partial u} = (1, v, 0), \qquad \frac{\partial \underline{r}(u,v)}{\partial v} = (0, u, 2v)$$

$$\frac{\partial \underline{r}(u,v)}{\partial u} \times \frac{\partial \underline{r}(u,v)}{\partial v} = \left| \begin{array}{ccc} \underline{i} & \underline{j} & \underline{k} \\ 1 & v & 0 \\ 0 & u & 2v \end{array} \right| \begin{array}{cc} \underline{i} & \underline{j} \\ 1 & v \\ 0 & u \end{array} =$$

$$= 2 \cdot v^2 \cdot \underline{i} + 0 \cdot \underline{j} + u \cdot \underline{k} - (0 \cdot \underline{k} + 0 \cdot \underline{i} + 2 \cdot v \cdot \underline{j}) = 2v^2 \underline{i} - 2v \underline{j} + u \underline{k} = \begin{pmatrix} 2v^2 \\ -2v \\ u \end{pmatrix}$$

$$\underline{v}(u, uv, v^2) = \begin{pmatrix} u \cdot uv \\ (v^2)^2 \\ uv \end{pmatrix} = \begin{pmatrix} u^2 v \\ v^4 \\ uv \end{pmatrix}$$

$$(\star) = \iint_D \begin{pmatrix} u^2v \\ v^4 \\ uv \end{pmatrix} \bullet \begin{pmatrix} 2v^2 \\ -2v \\ u \end{pmatrix} \, \mathrm{d}u\mathrm{d}v = \iint_D 2u^2v^3 - 2v^5 + u^2v \, \mathrm{d}u\mathrm{d}v =$$

$$= \int_{u=-1}^{2} \left(\int_{v=0}^{1} 2u^2v^3 - 2v^5 + u^2v \, \mathrm{d}v \right) \, \mathrm{d}u =$$

$$= \int_{u=-1}^{2} \left[2u^2\frac{v^4}{4} - 2\frac{v^6}{6} + u^2\frac{v^2}{2} \right]_{v=0}^{1} \, \mathrm{d}u =$$

$$= \int_{u=-1}^{2} \left[\frac{1}{2}u^2v^4 - \frac{1}{3}v^6 + \frac{1}{2}u^2v^2 \right]_{v=0}^{1} \, \mathrm{d}u = \int_{u=-1}^{2} \frac{1}{2}u^2 - \frac{1}{3} + \frac{1}{2}u^2 - 0 \, \mathrm{d}u =$$

$$= \int_{u=-1}^{2} u^2 - \frac{1}{3} \, \mathrm{d}u = \left[\frac{u^3}{3} - \frac{1}{3}u \right]_{u=-1}^{2} = \frac{8}{3} - \frac{2}{3} - \left(\frac{-1}{3} + \frac{1}{3} \right) = 2$$

Exercise 5.4.3 *Calculate the line integral of* $\underline{v}(x, y, z) = (xy^2z, x^2yz, \frac{1}{2}x^2y^2)$ *if the curve is a circle given by* $x^2 + y^2 = 4,\ z = 0$.

Solution.

$$G: \quad \underline{r}(t) = (2 \cdot \cos t, 2 \cdot \sin t, 0) \quad t \in [0, 2\pi]$$

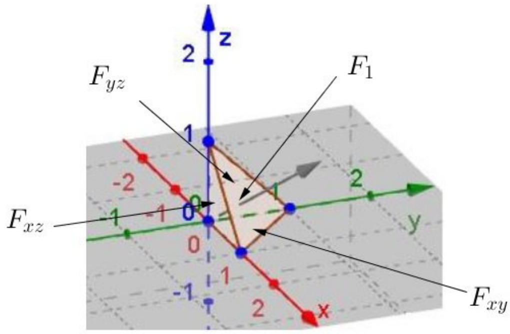
$\underline{r}'(t) = (-2 \cdot \sin t, 2 \cdot \cos t, 0)$

$$\underline{v}(\underline{r}(t)) = \underline{v}(2 \cdot \cos t, 2 \cdot \sin t, 0) = (0, 0, \frac{1}{2} \cdot 4 \cdot \cos^2 t \cdot 4 \cdot \sin^2 t) =$$

$$= (0, 0, 8 \cdot \cos^2 t \cdot \sin^2 t)$$

$$\int_G \underline{v}(\underline{r}) \, \mathrm{d}\underline{r} = \int_{t_1}^{t_2} \underline{v}(\underline{r}(t)) \bullet \underline{r}'(t) \, \mathrm{d}t = \int_0^{2\pi} 0 \, \mathrm{d}t = 0.$$

Exercise 5.4.4 *Calculate the integrals in the Divergence theorem if* $\underline{v}(x, y, z) = (xy, zx, yz)$ *and the body enclosed by the planes* $x = 0,\ y = 0,\ z = 0$ *and* $x + y + z = 1$.



Solution.

$$\int_F \underline{v}\,\mathrm{d}f = \int_{F_1} \underline{v}\,\mathrm{d}f + \int_{F_{xz}} \underline{v}\,\mathrm{d}f + \int_{F_{yz}} \underline{v}\,\mathrm{d}f + \int_{F_{xy}} \underline{v}\,\mathrm{d}f$$

- $F_1: \quad \underline{r}_1(u,v) = (u, v, 1-u-v)\ D_1 = \{(u,v): \quad u \in [0,1], 0 \le v \le 1-u\}$

$$\frac{\partial \underline{r}_1}{\partial u}(u,v) = (1,0,-1) \qquad \frac{\partial \underline{r}_1}{\partial v}(u,v) = (0,1,-1)$$

$$\frac{\partial \underline{r}_1(u,v)}{\partial u} \times \frac{\partial \underline{r}_1(u,v)}{\partial v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1,1,1)$$

$$\underline{v}(\underline{r}_1(u,v)) = (uv, (1-u-v)u, v(1-u-v))$$

- $F_{xz}: \quad \underline{r}_{xz}(u,v) = (u,0,v)\ D_{xz} = \{(u,v): \quad u \in [0,1], 0 \le v \le 1-u\}$

$$\underline{v}(\underline{r}_{xz}(u,v)) = (0, uv, 0)$$

- $F_{yz}: \quad \underline{r}_{yz}(u,v) = (0,u,v)\ D_{yz} = \{(u,v): \quad u \in [0,1], 0 \le v \le 1-u\}$

$$\underline{v}(\underline{r}_{yz}(u,v)) = (0,0,uv)$$

- $F_{xy}: \quad \underline{r}_{yz}(u,v) = (u,v,0)\ D_{xy} = \{(u,v): \quad u \in [0,1], 0 \le v \le 1-u\}$

$$\underline{v}(\underline{r}_{xy}(u,v)) = (uv,0,0)$$

$\int_F \underline{v}\,\mathrm{d}f = \iint_{D_1} \underline{v}(\underline{r}_1(u,v)) \bullet (1,1,1)\ \mathrm{d}u\mathrm{d}v + \iint_{D_{xz}} \underline{v}(\underline{r}_{xz}(u,v)) \bullet (0,-1,0)\ \mathrm{d}u\mathrm{d}v + \iint_{D_{yz}} \underline{v}(\underline{r}_{yz}(u,v)) \bullet (-1,0,0)\ \mathrm{d}u\mathrm{d}v + \iint_{D_{xy}} \underline{v}(\underline{r}_{xy}(u,v)) \bullet (0,0,-1)\ \mathrm{d}u\mathrm{d}v$

$$\int_F \underline{v}\ \mathrm{d}f = \int_{u=0}^{1}\int_{v=0}^{1-u} (uv, u-u^2-uv, v-uv-v^2) \bullet (1,1,1)\ \mathrm{d}u\mathrm{d}v +$$

$$+\int_{u=0}^{1}\int_{v=0}^{1-u} (0, uv, 0) \bullet (0,-1,0)\ \mathrm{d}u\mathrm{d}v +$$

$$+\int_{u=0}^{1}\int_{v=0}^{1-u} (0, 0, uv) \bullet (-1,0,0)\ \mathrm{d}u\mathrm{d}v +$$

$$+\int_{u=0}^{1}\int_{v=0}^{1-u} (uv, 0, 0) \bullet (0,0,-1)\ \mathrm{d}u\mathrm{d}v$$

$$\int_F \underline{v}\ \mathrm{d}f = \int_{u=0}^{1}\int_{v=0}^{1-u} uv + u - u^2 - uv + v - uv - v^2\ \mathrm{d}u\mathrm{d}v +$$

$$+\int_{u=0}^{1}\int_{v=0}^{1-u} -uv\ \mathrm{d}u\mathrm{d}v + 0 + 0 =$$

$$= \int_{u=0}^{1}\int_{v=0}^{1-u} (u+v) - (u^2+2uv+v^2)\ \mathrm{d}u\mathrm{d}v =$$

$$= \int_{u=0}^{1}\int_{v=0}^{1-u} (u+v) - (u+v)^2\ \mathrm{d}u\mathrm{d}v =$$

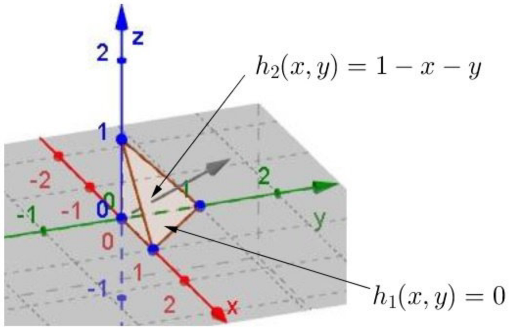
$$\int_{u=0}^{1} \left[\frac{(u+v)^2}{2\cdot 1} - \frac{(u+v)^3}{3\cdot 1}\right]_{v=0}^{1-u} \mathrm{d}u = \int_{u=0}^{1} \frac{1}{2} - \frac{1}{3} - \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\ \mathrm{d}u =$$

$$= \int_{u=0}^{1} \frac{1}{6} - \frac{1}{2}u^2 + \frac{1}{3}u^3\ \mathrm{d}u = \left[\frac{1}{6}u - \frac{1}{6}u^3 + \frac{1}{12}u^4\right]_{u=0}^{1} = \frac{1}{12}$$

Now we calculate the right hand side of the Divergence theorem:

$$div\underline{v}(x,y,z) = \partial_1 v_1(x,y,z) + \partial_2 v_2(x,y,z) + \partial_3 v_3(x,y,z) = y + 0 + y = 2y$$

$$D = \{(x,y) : \quad x \in [0,1], 0 \le y \le 1-x\}$$



On the other hand the body can be given as a normal domain in the space (enclosed by surfaces), thus

$$\int_V div\underline{v}\ \mathrm{d}V = \int\int_D \left(\int_{h_1(x,y)}^{h_2(x,y)} div\underline{v}(x,y,z)\,dz\right) dx\,dy =$$

$$\int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 2y\,dz\right) dy\right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} 2y(1-x-y)dy\right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} (2y-2xy-2y^2)dy\right) dx =$$

$$= \int_0^1 \left[y^2 - xy^2 - \frac{2}{3}y^3\right]_0^{1-x} dx =$$

$$= \int_0^1 (1-x)^2 - x(1-x)^2 - \frac{2}{3}(1-x)^3\,dx =$$

$$= \int_0^1 (1-x)^2(1-x) - \frac{2}{3}(1-x)^3\,dx = \int_0^1 \frac{1}{3}(1-x)^3\,dx =$$

$$= \frac{1}{3}\cdot\left[\frac{(1-x)^4}{4\cdot(-1)}\right]_0^1 = -\frac{1}{12}\cdot\left[(1-x)^4\right]_0^1 = -\frac{1}{12}\cdot(0-1) = \frac{1}{12}$$